Influence in Weighted Committees

Sascha Kurz
Dept. of Mathematics, University of Bayreuth
sascha.kurz@uni-bayreuth.de

Alexander Mayer
Dept. of Economics, University of Bayreuth
alexander.mayer@uni-bayreuth.de

Stefan Napel
Dept. of Economics, University of Bayreuth
stefan.napel@uni-bayreuth.de

October 16, 2020

Abstract

Committee decisions on more than two alternatives much depend on the adopted aggregation rule, and so does the distribution of power among committee members. We quantify how different voting methods such as pairwise majority votes, plurality voting with or without a runoff, or Borda rule map asymmetric numbers of seats, shares, voting weights, etc. to influence on collective outcomes when individual preferences vary. Generalizations of the Penrose-Banzhaf and Shapley-Shubik power indices are proposed and applied to elections of the IMF Managing Director. Previous analysis of a priori power in binary voting is thus extended to universal social choice rules.

Keywords: weighted voting · voting power · weighted committee games · plurality runoff · Borda rule · Copeland rule · IMF Executive Board · IMF Managing Director

We are grateful to Hannu Nurmi for stimulating discussions and to two anonymous reviewers for many constructive comments. We also benefited from feedback on presentations in Cardiff, Dagstuhl, Graz, Hagen, Hamburg, Heidelberg, Jerusalem, Leipzig, Moscow, St. Etienne and Turku.

1 Introduction

The aggregation of individual preferences by some form of voting is commonplace in politics, business, and everyday life. People are rarely aware, however, of how much collective choices can vary with the adopted aggregation rule. For illustration imagine a hiring committee that comprises three groups with six, five, and three members each. Suppose they have strict preference relations P_i, $i \in \{1, 2, 3\}$, over five applicants, a, b, c, d, and e, that rank these in the following decreasing orders: $aP_1dP_1eP_1cP_1b$ for members of group 1, $bP_2cP_2dP_2eP_2a$ for group 2, and $cP_3eP_3dP_3bP_3a$ for group 3. If everyone votes sincerely according to these preferences (for informational or institutional reasons) then a receives the position under *plurality rule* with 6 vs. 5 vs. 3 votes for a, b, and c. A *runoff vote* between the plurality leaders a and b, given that neither is supported by a majority, would make b the winner with a count of $8:6$. Candidate c, however, beats b and any other candidate in *pairwise majority votes*. The *Borda scoring rule* singles out d as winner. And candidate e could win if the *approval voting method* is applied.[1] In other words, the hiring choice is entirely up to which voting rule is used.

With enough information about preferences, each group might work out their 'ideal' voting method for the decision at hand: group 1 could try to impose plurality rule in order to have its way, or group 3 might argue for pairwise comparisons. But voting rules are often adopted in advance and for many decisions, not case by case. The question this paper seeks to address is therefore: how does adoption of one aggregation method rather than another affect a group's success or influence a priori, i.e., not yet knowing what will be the applicable preferences?

Can we say if small groups are generally enjoying greater leverage when committees fill a position, elect an official, or select a motion by plurality or pairwise votes? Which rules from a given list of suggestions maximize (or minimize) the expected influence of a particularly sized group in a committee, and which rule involves the smallest misalignment between applicable voting weights and induced distribution of influence? There is a huge literature on voting power but these questions have to our knowledge not been addressed yet. We aim to change this. The goal is to quantify how different aggregation methods such as plurality with or without a runoff, Borda count, or Copeland rule map asymmetric voting weights to influence on outcomes

[1] Approval voting entails: (i) each voter indicates approval for an arbitrary subset of candidates (e.g., group 1: $\{a, d, e\}$; group 2: $\{b, c\}$; group 3: $\{c, e\}$) ; (ii) the candidate with the most approvals wins (6+3 for e). Formal definitions of the voting rules that we investigate will be given in Section 3.1 below.

when individual preferences vary.

We build on tools that were developed for analysis of simple voting games with binary 'yes'-or-'no'options. Namely, the *Penrose-Banzhaf index* (Penrose 1946; Banzhaf 1965) and the *Shapley-Shubik index* (Shapley and Shubik 1954) are prominent indicators of voting power. They evaluate sensitivity of the collective choice to changes in a given voter's preferences. This sensitivity is operationalized as the likelihood of the voter being *pivotal* or *critical*: flipping the individual vote would flip the collective decision. Applications range widely and include, e.g., the US Electoral College, the EU Council of Ministers and the IMF Executive Board.

This paper extends voting power analysis to *weighted committees*. These are tuples $(N, A, r|\mathbf{w})$ that specify a set $N = \{1, \ldots, n\}$ of players, a set $A = \{a_1, \ldots, a_m\}$ of collective decision alternatives, and the combination $r|\mathbf{w}$ of an anonymous voting method r (e.g., plurality rule, Borda rule, and so on) with a vector $\mathbf{w}$ of integers that represent group sizes, voting shares, etc. High influence or voting power of player i in a committee is deduced from high sensitivity of the outcome to i's preferences.

More precisely, influence of a player $i \in N$ is measured either as the probability for a random change in i's preferences causing a change of the collective choice, or by how i's probability relates to the one that would correspond to being a dictator. We focus on the case in which all profiles of strict preference orderings are assumed to be equally likely a priori. This generalizes the Penrose-Banzhaf index for binary voting games to social choice from $m \geq 3$ alternatives. An alternative probability assumption generalizes the Shapley-Shubik index. The respective influence indications can identify winners and losers of institutional reform. They help stakeholders, lobbyists and others to assess the distribution of power and possibly to change the adopted aggregation rule to their advantage.

2 Related Work

The power distribution in binary weighted voting games has been of interest at least since von Neumann and Morgenstern (1953, Ch. 10) formalized these as a subclass of so-called *simple (voting) games*. See Felsenthal and Machover (1998), Laruelle and Valenciano (2008) or Napel (2019) for overviews. The common binary framework can be restrictive, however. Even for collective 'yes'-or-'no' decisions, individual voters usually have more than two options. They can abstain or not attend a vote, which may affect the outcome differently than casting a vote either way. Corresponding situations have been formalized as *ternary voting games* (Felsenthal and Machover 1997; Tchantcho et al. 2008; Parker 2012) and *quaternary voting games* (Laruelle and Valenciano 2012). Players may also express graded intensities of support: in (j, k) *simple games*, studied by Hsiao and Raghavan (1993) and Freixas and Zwicker (2003, 2009), each player selects one of j ordered levels of approval. The resulting partitions of players are mapped to one of k ordered output levels; respective power indices have been defined by Freixas (2005a, 2005b).

Linear orderings of actions and feasible outcomes, as required by (j, k) simple games, are given naturally in many applications but fail to exist in others – especially when candidates for office, policy programs, locations of a facility, etc. have multidimensional attributes. Pertinent extensions of simple games, along with corresponding power measures, have been introduced as *multicandidate voting games* by Bolger (1986) and taken up as *simple r-games* by Amer, Carreras, and Magãna (1998) and as *weighted plurality games* by Chua, Ueng, and Huang (2002). They require players to each cast their votes for one of r candidates.

We here draw on the yet more general framework of *weighted committee games* (Kurz, Mayer, and Napel 2020): winners can depend on the entire profile of preference rankings of voters rather than just top elements. We then conceive of *player i's influence* or *voting power* as the sensitivity of joint decisions to i's preferences. The resulting ability to affect collective outcomes is closely linked to the opportunity to manipulate social choices in the sense of Gibbard (1973) and Satterthwaite (1975). Our investigation therefore relates to computational studies by Nitzan (1985), Kelly (1993), Aleskerov and Kurbanov (1999), or Smith (1999) that have quantified the aggregate *manipulability* of a given decision rule. The conceptual difference between manipulability indices and the power indices defined below is that we evaluate consequences of arbitrary preference perturbations, while the indicated studies consider preference misrepresentation that is beneficial from the perspective of a player's original preferences.[2] Voting power could be used to a player's strategic advantage but it need not. A preference change might result from log-rolling or external lobbying (where costs of persuasion can relate more to preference intensity than a player's original ranking), or could be a demonstration of power for its own sake.

[2]Nitzan (1985) also checked if outcomes could be affected by arbitrary variations of preferences and tracked this at the aggregate level. We break the latter down to individual players and link outcome sensitivity to voting weights.

Rule	*Winning alternative at preference profile* $\mathbf{P}$
Borda	$r^B(\mathbf{P}) \in \arg\max_{a \in A} \sum_{i \in N} b_i(a, \mathbf{P})$
Copeland	$r^C(\mathbf{P}) \in \arg\max_{a \in A} \left\|\{a' \in A \mid a \succ^{\mathbf{P}}_M a'\}\right\|$
Plurality	$r^P(\mathbf{P}) \in \arg\max_{a \in A} \left\|\{i \in N \mid \forall a' \neq a \in A \colon aP_ia'\}\right\|$
Plurality runoff	$r^{PR}(\mathbf{P}) \begin{cases} = r^P(\mathbf{P}) \text{ if } \left\|\{i \in N \mid \forall a' \in A \setminus \{r^P(\mathbf{P})\} \colon r^P(\mathbf{P})P_ia'\}\right\| > \frac{n}{2}, \text{else} \\ \in \underset{a \in \{a_{(1)}, a_{(2)}\}}{\arg\max} \left\|\{i \in N \mid \forall a' \neq a \in \{a_{(1)}, a_{(2)}\} \colon aP_ia'\}\right\| \end{cases}$
Instant runoff	$r^{IR}(\mathbf{P}) \begin{cases} = r^P(\mathbf{P}) \text{ if } \left\|\{i \in N \mid \forall a' \in A \setminus \{r^P(\mathbf{P})\} \colon r^P(\mathbf{P})P_ia'\}\right\| > \frac{n}{2}, \text{else} \\ = r^{IR}(\tilde{\mathbf{P}}) \text{ deleting } \underset{a \in A}{\arg\min} \left\|\{i \in N \mid \forall a' \neq a \in A \colon aP_ia'\}\right\| \text{ from } \mathbf{P}, A \end{cases}$

Table 1: Investigated voting rules

3 Preliminaries

3.1 Voting Rules

We will consider a set $N = \{1, \ldots, n\}$ of voters or players such that each voter $i \in N$ has strict preferences P_i over a finite set $A = \{a_1, \ldots, a_m\}$ of $m \geq 2$ alternatives. We write abc in abbreviation of aP_ibP_ic when the player's identity is clear. The set of all $m!$ strict preference orderings on A is denoted by $\mathcal{P}(A)$. A *(resolute) voting rule* $r \colon \mathcal{P}(A)^n \to A$ maps each preference profile $\mathbf{P} = (P_1, \ldots, P_n)$ to a winning alternative $a^* = r(\mathbf{P})$. Rule r is *anonymous* if for any $\mathbf{P} \in \mathcal{P}(A)^n$ and any permutation $\sigma \colon N \to N$ we have $r(\mathbf{P}) = r(\sigma(\mathbf{P}))$ where $\sigma(\mathbf{P}) := (P_{\sigma(1)}, \ldots, P_{\sigma(n)})$.

We restrict attention to truthful voting[3] under one of the five anonymous rules summarized in Table 1, assuming lexicographic tie breaking. Our selection comprises two scoring rules (Borda, plurality), one Condorcet-consistent pairwise majority method (Copeland), and two multi-stage procedures that are in use for filling political offices in many European jurisdictions (plurality runoff and instant runoff). See Myerson (1999) or Laslier (2012) on properties and pros and cons of these and other voting procedures.

[3] In principle, the same kind of power analysis could be carried out also for strategic voters. This would require knowledge of the mapping from profiles of players' preferences to the element of A (or a probability distribution over A) which is induced by the selected voting equilibrium. Determination of the latter is in general highly non-trivial (see, e.g., Bouton 2013).

Under *plurality rule* r^P each voter simply names his or her top-ranked alternative and the alternative that is ranked first by the most voters is chosen. This is also the winner under *plurality (with) runoff rule* r^{PR} if the obtained plurality constitutes a majority (i.e., more than 50% of votes); otherwise a binary runoff vote between the alternatives $a_{(1)}$ and $a_{(2)}$ that obtained the highest and second-highest plurality scores in the first stage is conducted. *Instant runoff* r^{IR} is similar except that in case no alternative gets a majority, the alternative $a_{(m)}$ (respectively $a_{(m-1)}$, $a_{(m-2)}$ etc.) that obtained the lowest plurality score gets sequentially eliminated until one alternative achieves a majority. r^{PR} and r^{IR} are equivalent if $m = 3$ or $n = 3$ under sincere voting.

Borda rule r^B has each player i assign $m-1, m-2, \ldots, 0$ points to the alternative that he or she ranks first, second, etc. These points $b_i(a, \mathbf{P}) := \left|\{a' \in A \mid aP_ia'\}\right|$ equal the number of alternatives that i ranks below a. The alternative with the highest total number of points, known as Borda score, is selected.

Copeland rule r^C considers pairwise majority votes between all alternatives. They define the majority relation $a \succ_M^{\mathbf{P}} a' \;:\Leftrightarrow\; \left|\{i \in N \mid aP_ia'\}\right| > \left|\{i \in N \mid a'P_ia\}\right|$ and the alternative that beats the most others according to $\succ_M^{\mathbf{P}}$ is selected. Copeland rule is Condorcet consistent: if some alternative a is a Condorcet winner, i.e., beats all others, then $r^C(\mathbf{P}) = a$.

3.2 Weighted Committees

Anonymous voting rules, such as those described above, treat components P_i and P_j of a preference profile $\mathbf{P}$ symmetrically. Still, individual preferences often feed into a collective decision asymmetrically when a committee votes. For instance, stockholders have as many votes as they own shares, or political leaders cast bloc votes in proportion to party seats. The resulting mapping from preferences to outcomes is a combination of anonymous voting rule r with weights $w_1, \ldots, w_n \in \mathbb{N}_0$ for players $1, \ldots, n$ denoted by

$$r|\mathbf{w}(\mathbf{P}) := r([P_1]^{w_1}, [P_2]^{w_2}, \ldots, [P_n]^{w_n}) = r(\underbrace{P_1, \ldots, P_1}_{w_1 \text{ times}}, \underbrace{P_2, \ldots, P_2}_{w_2 \text{ times}}, \ldots, \underbrace{P_n, \ldots, P_n}_{w_n \text{ times}}) \qquad (1)$$

for all $\mathbf{P} \in \mathcal{P}(A)^n$. The combination $(N, A, r|\mathbf{w})$ of a set of voters, a set of alternatives and a particular weighted voting rule defines a *weighted committee (game)*. When the underlying rule is plurality rule r^P, then $(N, A, r^P|\mathbf{w})$ is called a *(weighted) plurality committee*. Similarly, $(N, A, r^{PR}|\mathbf{w})$, $(N, A, r^{IR}|\mathbf{w})$, $(N, A, r^B|\mathbf{w})$ and $(N, A, r^C|\mathbf{w})$ are referred

P_1	P_2	P_3	P_4
c	d	e	a
b	a	b	e
a	b	a	d
e	e	d	b
d	c	c	c

$\Rightarrow$

$r^B|\mathbf{w}(\mathbf{P}) = a$ (a has max. Borda score 36)
$r^C|\mathbf{w}(\mathbf{P}) = b$ (b has max. pairwise wins 4)
$r^P|\mathbf{w}(\mathbf{P}) = c$ (c has max. plurality tally 5)
$r^{PR}|\mathbf{w}(\mathbf{P}) = d$ (d beats c in runoff vote by $9:5$)
$r^{IR}|\mathbf{w}(\mathbf{P}) = e$ (deletion of $b \searrow a \searrow d$; e beats c by $9:5$)

Table 2: Choices for preference profile $\mathbf{P}$ when $\mathbf{w} = (5, 4, 3, 2)$

to as *plurality runoff*, *instant runoff*, *Borda* and *Copeland committees*.[4] These committees can all yield mutually distinct decisions for fixed $\mathbf{w}$ if $m > 2$. This is illustrated in Table 2 for $N = \{1, 2, 3, 4\}$, $A = \{a, b, c, d, e\}$, and $\mathbf{w} = (5, 4, 3, 2)$.

Weighted committees $(N, A, r|\mathbf{w})$ and $(N, A, r'|\mathbf{w}')$ are equivalent if the respective mappings from preference profiles to outcomes a^* coincide: $r|\mathbf{w}(\mathbf{P}) = r'|\mathbf{w}'(\mathbf{P})$ for all $\mathbf{P} \in \mathcal{P}(A)^n$. If there is no need to highlight the underlying rule r and weights $\mathbf{w}$, we will denote the respective mapping from preference profiles to outcomes by $\rho \equiv r|\mathbf{w}$ and refer to *committee (game)* (N, A, ρ). Ω denotes the set of all committee games.

Binary committees (N, A, ρ) for which $A = \{0, 1\}$ and ρ is surjective and monotonic correspond to *simple (voting) games*, which were introduced by von Neumann and Morgenstern (1953, Ch. 10). They are commonly described as a pair (N, v), where characteristic function $v\colon 2^N \to \{0, 1\}$ classifies each set $S \subseteq N$ of voters as either a *winning coalition* ($v(S) = 1$), meaning that support of members of S for $a = 1$ is sufficient to replace default alternative $a = 0$, or otherwise as a *losing coalition* ($v(S) = 0$). If a specific binary committee (N, A, ρ) is given, let (N, v_ρ) denote the respective simple game with $v_\rho(S^{\mathbf{P}}) = 1\ (0) \Leftrightarrow \rho(\mathbf{P}) = 1\ (0)$ where $S^{\mathbf{P}} := \{i \in N : 1P_i0\}$.

Two equivalent committees evidently come with identical expectations for individual players to influence the collective decision (voting power) and to obtain outcomes that match their own preferences (success). We will here focus on power and non-equivalent committees that either involve the same rule r but different weights $\mathbf{w}$ and $\mathbf{w}'$, or the same weights $\mathbf{w}$ but different rules r and r'. We seek to quantify: to what extent does a change of voting weights, implied for example by a reform of quotas in the International Monetary Fund or a member of parliament switching party, shift the respective balance of power? How does players' attractiveness to a lobbyist change when a committee replaces one voting method by another?

[4]See Kurz et al. (2020) on some of their structural properties.

4 Measuring Influence in Weighted Committees

4.1 Classical Power Indices and Probability Assumptions

Informal attempts to quantify the balance of power in binary committees date back to the Constitutional Convention in Philadelphia in 1787. See Riker (1986). In the first rigorous investigation, Penrose (1946) assumed strict preferences over two options to be equally likely and independent across voters. Then he studied the probability of a given voter $i \in N$ to be *pivotal* or *critical*: in the terminology of simple games, i is either able to turn a winning coalition $S \subseteq N$ with $i \in S$ into a losing one by leaving ($v(S) - v(S \setminus i) = 1$), or to turn a losing coalition $S \subseteq N \setminus i$ into a winning one by joining ($v(S \cup i) - v(S) = 1$).[5] The same idea was independently pursued by Banzhaf (1965) and

$$PBI_i(N, v) = \sum_{\substack{S \subseteq N, \\ i \notin S}} \frac{1}{2^{n-1}} [v(S \cup i) - v(S)] = \sum_{\substack{S \subseteq N, \\ i \in S}} \frac{1}{2^{n-1}} [v(S) - v(S \setminus i)], \quad i \in N \tag{2}$$

is today known as the *Penrose-Banzhaf index (PBI)* of simple game (N, v) with $n = |N|$.

Investigations of voting power in the EU Council, the US Electoral College, etc. (see, e.g., Holler and Nurmi 2013) typically use the PBI and its normalized version

$$nPBI_i(N, v) = \frac{PBI_i(N, v)}{\sum_{j \in N} PBI_j(N, v)}, \tag{3}$$

or the *Shapley-Shubik index (SSI)*

$$SSI_i(N, v) = \sum_{\substack{S \subseteq N, \\ i \notin S}} \frac{s! \cdot (n - s - 1)!}{n!} [v(S \cup i) - v(S)] = \sum_{\substack{S \subseteq N, \\ i \in S}} \frac{(s-1)! \cdot (n-s)!}{n!} [v(S) - v(S \setminus i)] \tag{4}$$

with $s = |S|$. The latter was introduced by Shapley and Shubik (1954) and specializes the Shapley value, which was defined for general cooperative games (N, v) with $v\colon 2^N \to \mathbb{R}$ by Shapley (1953), to simple games. See Felsenthal and Machover (1998), Laruelle and Valenciano (2008) or Napel (2019).

Formally, a power index ψ is a mapping that assigns a vector in $\mathbb{R}^n$ to every element of a given class of voting games, where ψ_i is indicating player i's voting power or influence in the respective game for a particular conception of influence. For one such

[5]For simplicity we will write $S \setminus i$ and $S \cup i$ instead of $S \setminus \{i\}$ and $S \cup \{i\}$, respectively.

conception, influence is ascribed to a player in proportion to the *sensitivity of collective decisions* to that player's preferences or behavior (see Napel and Widgrén 2004). In binary committees, this sensitivity reduces to events in which $v(S \cup i) - v(S) = 1$ or $v(S) - v(S \setminus i) = 1$. Power indices such as PBI and SSI then differ in the probability assumptions about players' preferences (or coalitions of players supporting $a = 1$) when evaluating these critical events.

In particular, the PBI in eq. (2) can also be written as summing $v(S \cup i) - v(S)$ and $v(S) - v(S \setminus i)$ over *all* $S \subseteq N$ weighted by $1/2^n$,

$$PBI_i(N, v) = \sum_{S \subseteq N} \frac{1}{2^n} [v(S \cup i) - v(S \setminus i)], \tag{5}$$

and hence equals the probability that player i's vote matters for the collective decision assuming an *impartial culture (IC)*. This assumption a priori takes preferences P_j to be independent random variables for all players $j \in N$ and each $P_j \in \mathcal{P}(A)$ to have equal probability. Analogously, the SSI in eq. (4) can be written as

$$SSI_i(N, v) = \sum_{S \subseteq N} \frac{s! \cdot (n-s)!}{(n+1)!} [v(S \cup i) - v(S \setminus i)] \tag{6}$$

(cf. Proposition 4 below) and equals the probability that the collective decision is sensitive to player i's vote under the *impartial anonymous culture (IAC)* assumption. IAC is symmetric or 'impartial' regarding all $\pi \in \mathcal{P}(A)$ just like IC. But it does not take all profiles $\mathbf{P} \in \mathcal{P}(A)^n$ to have the same probability. Rather, allowing general $m \geq 2$, IAC assumes all anonymous preference counts $\mathbf{n} = (n_1, \ldots, n_{m!}) \in \mathbb{N}_0^{m!}$ with $n_1 + \ldots + n_{m!} = n$, where n_k denotes the number of voters whose preferences equal the k-th element of $\mathcal{P}(A)$, to be equiprobable. In the binary case, IAC takes all numbers $s = 0, \ldots, n$ of players with $1P_i0$ to have probability $\frac{1}{n+1}$ and, for given s, all corresponding subsets $S \subseteq N$ to have probability $\binom{n}{s}^{-1}$. This is incompatible with statistical independence and reflects a particular degree of correlation between voter preferences (see Section 8).

It almost goes without saying that the distribution of preferences in any real-world committee likely differs from IC, IAC or any of the more general cultures that we will consider below.[6] Influence indications by PBI, SSI, and their generalizations are *a priori assessments* from behind a veil of ignorance. These – like simplifying thought experiments that involve 'veils of ignorance' in general – can help to assess

[6]See, e.g., Regenwetter, Grofman, Marley, and Tsetlin (2012, Ch. 1).

the playing field created by voting weights. They are also useful for comparative statics. But they must not be mistaken for actual (a posteriori) influence relations in an institution. The latter are presumably based not just on more complex preference structures but also on political and social dimensions of power that are unrelated to the applicable voting mechanism, which is our focus.

4.2 Influence as Expected Sensitivity of Committee Decisions

We apply the idea of measuring (a priori) influence as sensitivity of the collective decision to individual players in committees that decide on $m \geq 2$ alternatives $A = \{a_1, \ldots, a_m\}$. Presume that respective preference profiles are drawn according to some fixed probability measure on $\mathcal{P}(A)^n$, reflecting IC, IAC or other culture assumptions. Let $\Pr(\mathbf{P}) \in [0, 1]$ denote the pertinent probability of profile $\mathbf{P}$ being realized. In order to assess the voting power of player i in the committee, we perturb i's realized preferences P_i to $P_i' \neq P_i \in \mathcal{P}(A)$ at random and check if the individual preference change would affect the outcome. Specifically, writing $\mathbf{P} = (P_i, \mathbf{P}_{-i})$ with $\mathbf{P}_{-i} = (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n)$, we are interested in the behavior of function

$$\Delta\rho(\mathbf{P}; P_i') := \begin{cases} 1 & \text{if} \quad \rho(\mathbf{P}) \neq \rho(P_i', \mathbf{P}_{-i}), \\ 0 & \text{if} \quad \rho(\mathbf{P}) = \rho(P_i', \mathbf{P}_{-i}). \end{cases} \tag{7}$$

We stay agnostic about the precise source of perturbations: a switch from P_i to P_i' might reflect a spontaneous change of mind or intentional preference misrepresentation, e.g., because someone has bought i's vote. Variations might also be the result of log-rolling, mistakes, or of receiving individual last-minute information about some of the candidates. Our important premise is only that: a committee member's input to the collective decision process matters more, the more influential player i is in the committee and vice versa.

One can then quantify player i's a priori influence – and compare it to that of other players or for variations of voting rule ρ such as moving from $r|\mathbf{w}$ to some $r|\mathbf{w}'$ – by taking expectations of $\Delta\rho(\mathbf{P}; P_i')$ over all $(m!)^n$ conceivable preference profiles $\mathbf{P}$ and all $m! - 1$ possible perturbations of P_i at any given $\mathbf{P}$:

$$\widehat{\mathcal{I}}_i(N, A, \rho) := \mathbb{E}\Big[\Delta\rho(\mathbf{P}; P_i')\Big] = \sum_{\mathbf{P} \in \mathcal{P}(A)^n} \frac{\Pr(\mathbf{P})}{m! - 1} \sum_{P_i' \neq P_i \in \mathcal{P}(A)} \Delta\rho(\mathbf{P}; P_i'), \quad i \in N. \tag{8}$$

A value of $\widehat{\mathcal{I}}_i(N, A, \rho) = 0.25$, for example, signifies that 25% of i's preference variations would change the outcome. We will verify below that $\widehat{\mathcal{I}}$ coincides with PBI and SSI for $m = 2$ when $\Pr(\mathbf{P})$ reflects the IC and IAC assumption, respectively.

Player i's power $\widehat{\mathcal{I}}_i(N, A, \rho)$ equals the probability that a change of i's preferences from P_i to a random $P_i' \neq P_i$ would affect the outcome when profiles $\mathbf{P}$ arise in a particular preference culture. If players' random preferences are exchangeable in that culture, $\widehat{\mathcal{I}}$ satisfies a natural *symmetry* condition: if $i, j \in N$ are symmetric players in $\Gamma = (N, A, \rho)$, i.e., $\rho(\mathbf{P}) = \rho(\sigma(\mathbf{P}))$ for all $\mathbf{P} \in \mathcal{P}(A)^n$ for a fixed permutation $\sigma \colon N \to N$ with $\sigma(i) = j$ and $\sigma(j) = i$, then $\widehat{\mathcal{I}}_i(\Gamma) = \widehat{\mathcal{I}}_j(\Gamma)$. Moreover $\widehat{\mathcal{I}}$ satisfies the so-called *null player property* of classical power indices: $\widehat{\mathcal{I}}_i(\Gamma) = 0$ if player i is a null player in Γ, i.e., its preferences never make a difference to the committee decision because ρ is constant in P_i.

$\widehat{\mathcal{I}}_i(N, A, \rho)$ generally falls short of one for a *dictator player*, i.e., when $\rho(\mathbf{P}) = a^*$ if and only if i ranks a^* top: since only changes of the dictator's top preference matter, only $(m! - (m-1)!)$ out of $m! - 1$ perturbations of P_i affect the outcome.[7] So maximal $\widehat{\mathcal{I}}(\cdot)$ numbers vary in m and it can be convenient to normalize the index such that indications always range from zero to one. This amounts to using $\mathcal{I}$ with

$$\mathcal{I}_i(N, A, \rho) := \frac{\widehat{\mathcal{I}}_i(N, A, \rho)}{(m! - (m-1)!)/(m! - 1)} = \sum_{\mathbf{P} \in \mathcal{P}(A)^n} \frac{\Pr(\mathbf{P})}{m! - (m-1)!} \sum_{P_i' \neq P_i \in \mathcal{P}(A)} \Delta\rho(\mathbf{P}; P_i'), \quad i \in N, \tag{9}$$

as a concise measure of *player i's a priori influence* or *voting power* in (weighted) committee (N, A, ρ).

Clearly, $\widehat{\mathcal{I}}_i(N, A, \rho) = \mathcal{I}_i(N, A, \rho)$ if $m = 2$. For $m \geq 2$ the normalization gives up $\widehat{\mathcal{I}}_i(N, A, \rho)$'s interpretation as a probability in favor of direct comparability across different committees:[8] regardless of how many alternatives or players are involved, $\mathcal{I}_i(N, A, \rho) \in [0, 1]$ quantifies how close i is to being a dictator in (N, A, ρ). $\mathcal{I}_i(N, A, \rho) = 0.5$, for instance, states that i's influence lies halfway between that of a null player and a dictator: on average, outcomes are half as sensitive to i's preferences than they would if i commanded all votes.

[7] Namely, $(m-1)! - 1$ preference perturbations leave the top rank unchanged.

[8] In contrast to the normalization in (3), which voids a meaningful probability interpretation in favor of *relative* power indications, $\widehat{\mathcal{I}}$ and $\mathcal{I}$ remain in one-to-one correspondence for given m.

5 Influence in Committees with an Impartial Culture

5.1 Generalization of PBI to Weighted Committees

Under the impartial culture assumption, which takes players' preferences $P_1, \ldots, P_n \in \mathcal{P}(A)$ to be independent and equiprobable, definitions (8) and (9) specialize to

$$\widehat{\mathcal{PBI}}_i(N, A, \rho) := \frac{\sum_{\mathbf{P} \in \mathcal{P}(A)^n} \sum_{P'_i \neq P_i \in \mathcal{P}(A)} \Delta\rho(\mathbf{P}; P'_i)}{(m!)^n \cdot (m! - 1)} \tag{10}$$

and

$$\mathcal{PBI}_i(N, A, \rho) := \frac{\sum_{\mathbf{P} \in \mathcal{P}(A)^n} \sum_{P'_i \neq P_i \in \mathcal{P}(A)} \Delta\rho(\mathbf{P}; P'_i)}{(m!)^n \cdot (m! - (m-1)!)}, \quad i \in N. \tag{11}$$

Both indices coincide with the classical Penrose-Banzhaf index PBI if $m = 2$.

Proposition 1. *Let $|A| = 2$. Then*

$$\mathcal{PBI}(N, A, \rho) = \widehat{\mathcal{PBI}}(N, A, \rho) = PBI(N, v_\rho).$$

Proof. The bijective relation between profiles $\mathbf{P}$ and coalitions $S^{\mathbf{P}} = \{j \in N : 1P_j0\}$ implies for all $i \in N$

$$\begin{aligned}
PBI_i(N, v_\rho) &= \frac{1}{2} \cdot \left(\frac{1}{2^{n-1}} \sum_{\substack{S \subseteq N, \\ i \notin S}} [v_\rho(S \cup i) - v_\rho(S)] + \frac{1}{2^{n-1}} \sum_{\substack{S \subseteq N, \\ i \in S}} [v_\rho(S) - v_\rho(S \setminus i)] \right) \qquad (12) \\
&= \frac{1}{2^n} \left(\sum_{\substack{\mathbf{P} \in \mathcal{P}(A)^n, \\ 0P_i1}} [v_\rho(S^{\mathbf{P}} \cup i) - v_\rho(S^{\mathbf{P}})] + \sum_{\substack{\mathbf{P} \in \mathcal{P}(A)^n, \\ 1P_i0}} [v_\rho(S^{\mathbf{P}}) - v_\rho(S^{\mathbf{P}} \setminus i)] \right) \\
&= \frac{1}{2^n} \left| \left\{ \mathbf{P} \in \mathcal{P}(A)^n : P'_i \neq P_i \Leftrightarrow \Delta\rho(\mathbf{P}, P'_i) = 1 \right\} \right| = \widehat{\mathcal{PBI}}_i(N, A, \rho) = \mathcal{PBI}_i(N, A, \rho).
\end{aligned}$$

■

$\mathcal{PBI}(N, A, \rho)$ and $\widehat{\mathcal{PBI}}(N, A, \rho)$ are not the only power indices that can be defined on the space Ω of committee games (N, A, ρ) to coincide with $PBI(N, v_\rho)$ for $m = 2$. For instance, the sum $\sum_{P'_i \neq P_i} \Delta\rho(\mathbf{P}; P'_i)$ in (10) and (11) could be replaced by an indicator function $\theta(\mathbf{P}; i)$ that equals 1 if $\Delta\rho(\mathbf{P}; P'_i) = 1$ for *some* $P'_i \neq P_i \in \mathcal{P}(A)$ and 0 otherwise; or by a count $\alpha(\mathbf{P}; i) = \left| \left\{ a \in A : \rho(P'_i, \mathbf{P}_{-i}) = a \text{ for } P'_i \in \mathcal{P}(A) \right\} \right| - 1$ of the number of alternatives different from $\rho(\mathbf{P})$ that a perturbation of player i's preferences could induce. The resulting indications coincide for $m = 2$ but differ for many committees if $m \geq 3$. Such differences echo Aumann's (1987) comment on multiplicity of game-theoretic solution concepts:

> "Different solution concepts are like different indicators of an economy; different methods for calculating a price index; different maps (road, topo, political, geologic, etc., not to speak of scale or projection, etc.); . . . They depict or illuminate the situation from different angles; each one stresses certain aspects at the expense of others."

Our definitions (10) and (11) stress the sensitivity aspect of influence. One is scaled to make numbers directly interpretable as a probability, the other to facilitate comparisons for different m.

5.2 Characterization of $\widehat{\mathcal{PBI}}$ and $\mathcal{PBI}$ by a Potential Function

Many solution concepts and power indices have been given 'characterizations' by sets of formal properties or *axioms* in addition to probabilistic, strategic or epistemic justifications.[9] The respective characterizations of classical power indices mostly exploit the mathematical lattice structure of simple games. No direct adaptation of these approaches is possible for classes of games that do not have this structure – such as general committee games $\Gamma \in \Omega$ as well as important subclasses of simple games (e.g., *weighted* simple games, where $v(S) = 1 \Leftrightarrow \sum_{i \in S} w_i \geq q$ for fixed weights $\mathbf{w} \in \mathbb{R}^n_+$ and quota $q \in \mathbb{R}_{++}$).

It is possible, however, to generalize to $\widehat{\mathcal{PBI}}$ and $\mathcal{PBI}$ a characterization of the PBI that uses a different strategy. Namely, Dragan (1996) and Ortmann (1998) established that the PBI is the unique index ψ that (i) *distributes the swings*, i.e., sums to $1/2^n$ times the total number of swings in the given game (where for all $S \subseteq N$ both $v(S \cup i) - v(S) = 1$ and $v(S) - v(S \setminus i) = 1$ are counted as a 'swing' for i), and (ii) *admits a potential (function)*. The latter means that there exists a mapping Q that assigns a real number $Q(N, v)$ to every simple game (N, v) such that one can conceive of player i's power $\psi_i(N, v)$ as i's contribution to game (N, v) (or rather to its potential). Namely, Q is a potential function for an index ψ if and only if for all (N, v) and $i \in N$

$$\psi_i(N, v) = Q(N, v) - Q(N \setminus i, v). \tag{13}$$

[9] The characterizations often came with delay, however. The PBI was first axiomatized by Dubey and Shapley (1979). Simple voting games form no vector space and so the first axiomatization of the SSI was given by Dubey (1975) two decades after Shapley (1953) and Shapley and Shubik (1954).

Here $(N \setminus i, v) = (N \setminus i, v^{N\setminus i})$ denotes the *subgame* or *restriction* of (N, v) after i is removed from the set of active players with $v^{N\setminus i}(S) := v(S)$ for all $S \subseteq N \setminus i$. In particular,

$$PBI_i(N, v) = Q(N, v) - Q(N \setminus i, v) \;\text{ for }\; Q(N, v) := \frac{\left|\{S \subseteq N : v(S) = 1\}\right|}{2^{n-1}}. \tag{14}$$

Viewing a simple game (N, v) as a binary committee $(N, \{0, 1\}, \rho)$ where coalitions S collect players j who share preference $1P_j0$, the 'subgame' evaluated in (13) defines a new mapping ρ' from preference profiles $\mathbf{P}_{-i} \in \mathcal{P}(A)^{n-1}$ of players $N \setminus i$ to outcomes $\rho'(\mathbf{P}_{-i}) := \rho(P_i, \mathbf{P}_{-i})$ with $0P_i1$. For $m > 2$ alternatives and $m! > 2$ different possibilities for player i's preference, it is in general necessary to consider more than one conceivable subgame among players $j \in N \setminus i$. Namely, the respective new mapping $\rho' : \mathcal{P}(A)^{n-1} \to A$ could reflect different ways in which i's preferences might be fixed in the background and set the stage for the choice by $N \setminus i$. For any given committee $\Gamma = (N, A, \rho)$, we therefore define the *restriction* $\Gamma_\pi^{N\setminus i} := (N \setminus i, A, \rho_\pi^{N\setminus i})$ with respect to player $i \in N$ and preference $\pi \in \mathcal{P}(A)$ by

$$\rho_\pi^{N\setminus i}(\mathbf{P}') := \rho(\mathbf{P}) \text{ where } P_j = P'_j \text{ for } j \in N \setminus i \text{ and } P_i = \pi \tag{15}$$

for all $\mathbf{P}' \in \mathcal{P}(A)^{n-1}$.

Now extend the difference consideration in (13) to all such restrictions. Specifically, say that a power index ψ that maps committees $\Gamma = (N, A, \rho) \in \Omega$ to $\psi(\Gamma) \in \mathbb{R}^n$ *admits an average potential* if there exists a potential function $\mathcal{Q} : \Omega \to \mathbb{R}$ such that

$$\psi_i(\Gamma) = \mathcal{Q}(\Gamma) - \frac{1}{m!} \cdot \sum_{\pi \in \mathcal{P}(A)} \mathcal{Q}(\Gamma_\pi^{N\setminus i}) \tag{16}$$

for all $\Gamma \in \Omega$ and $i \in N$. Then one can conceive of $\psi_i(\Gamma)$ as reflecting player i's expected contribution to the committee Γ relative to all subgames – i.e., certain well-defined committees $(N \setminus i, A, \rho')$ – that do not involve i.[10] Writing

$$\chi_i(N, A, \rho) := \left|\left\{(\mathbf{P}, P'_i) \in \mathcal{P}(A)^{n+1} : \rho(\mathbf{P}) \neq \rho(P'_i, \mathbf{P}_{-i})\right\}\right| \tag{17}$$

for the *number of swings of player i* in $\Gamma = (N, A, \rho)$, we have:

[10] If (N, A, ρ) involves a neutral rule ρ, potentials of all subgames $\Gamma_\pi^{N\setminus i}$, $\pi \in \mathcal{P}(A)$, coincide. Then (16) reduces to $\psi_i(\Gamma) = \mathcal{Q}(\Gamma) - \mathcal{Q}(\Gamma_\pi^{N\setminus i})$ for arbitrary fixed $\pi \in \mathcal{P}(A)$, analogous to (13).

Proposition 2. *$\widehat{\mathcal{PBI}}$ admits an average potential. The respective potential function is*

$$\mathcal{Q}(\Gamma) := \frac{1}{(m!-1)\cdot(m!)^n}\sum_{j\in N}\chi_j(\Gamma).$$

Proof. We have $\mathcal{Q}(\Gamma_\pi^{\varnothing}) = 0$ and for $n \geq 1$

$$\begin{aligned}\sum_{\pi\in\mathcal{P}(A)} \mathcal{Q}(\Gamma_\pi^{N\setminus i}) &= \sum_{\pi\in\mathcal{P}(A)} \frac{1}{(m!-1)\cdot(m!)^{n-1}}\sum_{j\in N\setminus i}\chi_j(\Gamma_\pi^{N\setminus i}) \\ &= \frac{1}{(m!-1)\cdot(m!)^{n-1}}\sum_{j\in N\setminus i}\sum_{\pi\in\mathcal{P}(A)}\sum_{\substack{\mathbf{P}\in\mathcal{P}(A)^n,\\ P_i=\pi}}\sum_{P_j'\neq P_j\in\mathcal{P}(A)}\Delta\rho(\mathbf{P},P_j') \\ &= \frac{1}{(m!-1)\cdot(m!)^{n-1}}\sum_{j\in N\setminus i}\chi_j(\Gamma).\end{aligned} \tag{18}$$

Hence

$$\begin{aligned}\mathcal{Q}(\Gamma) - \frac{1}{m!}\cdot\sum_{\pi\in\mathcal{P}(A)}\mathcal{Q}(\Gamma_\pi^{N\setminus i}) &= \frac{1}{(m!-1)\cdot(m!)^n}\sum_{j\in N}\chi_j(\Gamma) - \frac{1}{(m!-1)\cdot(m!)^n}\sum_{j\in N\setminus i}\chi_j(\Gamma) \\ &= \frac{\chi_i(\Gamma)}{(m!-1)\cdot(m!)^n} = \widehat{\mathcal{PBI}}_i(\Gamma).\end{aligned} \tag{19}$$

∎

Denote the total number of all players' swings divided by the (player-independent) number $(m!)^n(m!-1)$ of all $(\mathbf{P},P_i') \in \mathcal{P}(A)^{n+1}$ s.t. $P_i \neq P_i'$ as

$$\mathcal{X}(\Gamma) := \frac{\sum_{i\in N}\chi_i(\Gamma)}{(m!)^n(m!-1)}. \tag{20}$$

Then, as the direct equivalent of Ortmann's (1998) corresponding property for simple games, we say an index ψ *distributes the swings* in committee games if

$$\sum_{i\in N}\psi_i(\Gamma) = \mathcal{X}(\Gamma). \tag{21}$$

Proposition 3. *$\widehat{\mathcal{PBI}}$ is the unique index that admits an average potential and distributes the swings in committee games.*

Proof. Let ψ be an arbitrary index that distributes the swings and admits an average potential. Let the respective potential function be $\widetilde{\mathcal{Q}}$ with $\widetilde{\mathcal{Q}}(\varnothing, A, \rho) \equiv 0$ (w.l.o.g.).

Then (16) and (21) imply

$$\mathcal{X}(\Gamma) = \sum_{i \in N} \psi_i(\Gamma) = n \cdot \widetilde{Q}(\Gamma) - \frac{1}{m!} \sum_{i \in N} \sum_{\pi \in \mathcal{P}(A)} \widetilde{Q}(\Gamma_\pi^{N \setminus i}) \tag{22}$$

or, equivalently,

$$\widetilde{Q}(\Gamma) = \frac{1}{n} \left(\mathcal{X}(\Gamma) + \frac{1}{m!} \sum_{i \in N} \sum_{\pi \in \mathcal{P}(A)} \widetilde{Q}(\Gamma_\pi^{N \setminus i}) \right) \tag{23}$$

for all $\Gamma = (N, A, \rho)$ involving $n \geq 1$ players and $m \geq 2$ alternatives. Doing induction on n, all values of $\widetilde{Q}$ on Ω can be recursively computed via (23). So $\widetilde{Q}$ and hence ψ are uniquely determined for all $\Gamma \in \Omega$. Now it remains to recall from Proposition 2 that $\widehat{\mathcal{PBI}}$ admits an average potential and distributes the swings. ■

Index $\mathcal{PBI}$ rescales $\widehat{\mathcal{PBI}}$ by a constant factor. So $\mathcal{PBI}$ can be characterized in perfect analogy to $\widehat{\mathcal{PBI}}$: rescale $\mathcal{X}(\Gamma)$ in (20) to $\mathcal{X}'(\Gamma) = \frac{(m!-1)\mathcal{X}(\Gamma)}{m!-(m-1)!}$ and use $\mathcal{X}'(\Gamma)$ in (21). The respective potential function is $Q'(\Gamma) = \frac{1}{(m!-(m-1)!)\cdot(m!)^n} \sum_{j \in N} \chi_j(\Gamma)$.

6 Illustration

6.1 A Toy Example

Let us evaluate the distribution of voting power with $\mathcal{PBI}$ when the stylized hiring committee with three homogeneous groups of 6, 5, and 3 members (see Introduction) adopts Borda rule r^B, that is weighted committee $(N, A, r^B|(6,5,3))$. With $|A| = 2$ candidates, the applicant ranked first by any two groups wins. So all three players are symmetric. They have power $(1/2, 1/2, 1/2)$ according to the classical Penrose-Banzhaf index, and $(1/3, 1/3, 1/3)$ according to the Shapley-Shubik index.

The symmetry is broken when three or more candidates are involved. Given $A = \{a, b, c\}$, $\mathcal{PBI}(N, A, r^B|(6,5,3))$ evaluates if a change of i's preference P_i makes a difference to the Borda winner for all $(3!)^3 = 216$ strict preference profiles $\mathbf{P} \in \mathcal{P}(A)^3$. Table 3 illustrates this for profile $\mathbf{P} = (bca, abc, cba)$. The Borda winner b at $\mathbf{P}$ has a score of $20 = 6 \cdot 2 + 5 \cdot 1 + 3 \cdot 1$ vs. 10 for a vs. 12 for c (first block of table). When preferences $P_1 = bca$ of group 1 are varied (second block), changes to $P'_1 \in \{abc, acb, cab, cba\}$ result in a new Borda winner (indicated by an asterisk) while $P'_1 = bac$ does not. Similarly, three out of five perturbations P'_2 of player 2's preferences would change the outcome (third block); but no variation of P_3 affects the committee choice (last block). The

$\mathbf{P}=$			P'_1	a	b	c	P'_2	a	b	c	P'_3	a	b	c
(bca,	abc,	cba)	abc	**22***	14	6	-	-	-	-	abc	16	**20**	6
	$\Downarrow$		acb	**22***	8	12	acb	10	15	**17***	acb	16	**17**	9
a	b	c	bac	16	**20**	6	bac	5	**25**	12	bac	13	**23**	6
10	**20**	12	-	-	-	-	bca	0	**25**	17	bca	10	**23**	9
			cab	16	8	**18***	cab	5	15	**22***	cab	13	**17**	12
			cba	10	14	**18***	cba	0	20	**22***	-	-	-	-

Table 3: Effect of perturbation of $\mathbf{P} = (bca, abc, cba)$ to $(P'_i, \mathbf{P}_{-i})$ on Borda scores

	$m=3$	$m=4$	$m=5$
$\mathcal{PBI}(r^P \vert (6,5,3))$	(0.6667, 0.4444, 0.4444)	(0.7500, 0.3750, 0.3750)	(0.8000, 0.3200, 0.3200)
$\mathcal{PBI}(r^{PR} \vert (6,5,3))$	(0.5556, 0.5556, 0.5000)	(0.5833, 0.5833, 0.5000)	(0.6000, 0.6000, 0.5000)
$\mathcal{PBI}(r^{IR} \vert (6,5,3))$	(0.5556, 0.5556, 0.5000)	(0.5833, 0.5833, 0.5000)	(0.6000, 0.6000, 0.5000)
$\mathcal{PBI}(r^B \vert (6,5,3))$	(0.6806, 0.5972, 0.3611)	(0.7372, 0.6246, 0.3644)	(0.7631, 0.6462, 0.3839)
$\mathcal{PBI}(r^C \vert (6,5,3))$	(0.5509, 0.5509, 0.5509)	(0.5851, 0.5851, 0.5851)	(0.6098, 0.6098, 0.6098)

Table 4: Voting power in committee $(N, A, r|(6,5,3))$ for $|A| = m$ and $r \in \{r^P, r^{PR}, r^{IR}, r^B, r^C\}$

considered profile $\mathbf{P} = (bca, abc, cba)$ therefore contributes $(^4/_{864}, ^3/_{864}, 0)$ to $\mathcal{PBI}(\cdot)$.

Aggregating the corresponding numbers for all $\mathbf{P} \in \mathcal{P}(A)^n$ yields

$$\mathcal{PBI}(N, A, r^B|(6,5,3)) = (^{588}/_{864}, ^{516}/_{864}, ^{312}/_{864}) \approx (0.6806, 0.5972, 0.3611). \tag{24}$$

So for the weights at hand, group 1 has almost 70% of the opportunities to swing the collective choice that it would have when deciding alone. This figure is roughly halved for group 3. A traditional power index like the PBI fails to pick up that symmetry of the groups pertains only to binary votes and could yield quite misleading results when an institution decides on more alternatives. One can also see from the numbers in (24) that $\mathcal{PBI}(N, A, \rho)$ need not add to one: the collective outcome at a given $\mathbf{P}$ may be sensitive to the preferences of several players at the same time, or to those of none.[11]

Of course, manual computations as in Table 3 are tedious. It is not difficult, though, to evaluate $\mathcal{PBI}$ with a standard desktop computer for up to five alternatives; and to compare the respective distribution of voting power to that arising from other voting rules. Table 4 summarizes findings for $r \in \{r^B, r^C, r^{IR}, r^P, r^{PR}\}$. As

[11] This motivates use of *nPBI* (see eq. (3)) instead of *PBI* in many applications to binary committees.

the comparison between $m = 2$ and 3 showed already, voting power varies in the number of alternatives. Under plurality rule, for instance, player 1 is closer to having dictatorial influence, the more alternatives split the vote of players 2 and 3.[12]

6.2 Election of the IMF's Managing Director

The International Monetary Fund (IMF) constitutes a prominent real-world example of weighted voting. Power indices have been applied to it many times. See, e.g., Leech (2003), Alonso-Meijide and Bowles (2005), Aleskerov, Kalyagin, and Pogorelskiy (2008), or Leech and Leech (2013). We extend previous analysis to three alternatives and the election of the IMF's Managing Director by the Executive Board.

The Executive Board consists of 24 members whose voting weights reflect financial contributions to the IMF, so-called *quotas*. The six largest contributors – USA, China, Japan, Germany, France and the UK – and Saudi Arabia currently provide one Executive Director each. The remaining 182 member countries are grouped into seventeen constituencies. Each supplies one Executive Director who represents all group members and wields their combined voting rights.

Various changes to the distribution of quotas have taken place since the IMF's foundation in 1944. The most recent reform was agreed in 2010 and started to be implemented in 2016. A significant share of votes has shifted from the USA and Western Europe to emerging and developing countries. In particular, China's vote share has gone up to 6.1% (compared to 3.8% before). India's share increased to 2.6% (2.3%), Russia's to 2.6% (2.4%), Brazil's to 2.2% (1.7%) and Mexico's to 1.8% (1.5%). On the same occasion, the IMF modified the selection of its key representative, the Managing Director (currently: Kristalina Georgieva).

Prior to the reform, the election process was criticized as intransparent and undemocratic: the Managing Director used to be a European chosen in backroom negotiations with the US. The new process is advertised as "open, merit based, and transparent" (IMF Press Release 16/19): all Executive Directors and IMF Governors may nominate candidates. If the number of nominees is too big, a shortlist of three candidates is drawn up based on indications of support. From this shortlist the

[12] Corresponding $\widehat{\mathcal{PBI}}(\cdot)$ numbers can be obtained by multiplication with $\frac{m!-(m-1)!}{m!-1}$, i.e., $4/5$, $18/23$ or $96/119$ for $m = 3$, 4 or 5, respectively. $\mathcal{PBI}(N, \{a_1, \ldots, a_m\}, r^P | (6, 5, 3))$ converges to $(1, 0, 0)$ as $m \to \infty$. Comparative statics are more involved for the other rules: bigger m tends to raise the share of profiles $\mathbf{P}$ at which *some* perturbation of P_i affects the outcome but lowers the fraction of perturbations P_i' that do so both for i and a hypothetical dictator. The sum of these effects here increases $\mathcal{PBI}_i$ for all $i \in N$ and $r \in \{r^B, r^C, r^{IR}, r^{PR}\}$ but not in general.

new Managing Director is elected "by a majority of the votes cast" in the Executive Board.[13]

The IMF has neither publicly nor upon our request specified how a "majority of the votes cast" is to be achieved for three candidates. This creates procedural leeway. Its voting power effects can be quantified with the proposed index. We evaluate the influence implications of the procedural choice between (*i*) plurality rule (acknowledging that a 'plurality' is sometimes also called a 'relative majority'), (*ii*) plurality with a runoff if none of the three shortlisted candidates initially secures 50% of the votes, or (*iii*) pairwise votes à la Copeland. Our analysis falls under the caveats expressed at the end of Section 4.1. In particular, $\mathcal{PBI}$ maintains the statistical independence assumption of most earlier analysis of the IMF. It is intended to provide an a priori assessment of the playing field created by weights and how this depends on voting procedures, not an estimate of who wields how much influence on the next candidate choice given current alliances, economic ties, etc.

Influence figures in Table 5 are based on Monte Carlo simulation with sufficiently many iterations so that differences within rows are significant at ≥95% confidence.[14] 2016's increase of vote shares for emerging market economies has noticeably raised their voting power, no matter which aggregation rule we consider. This is most pronounced for China, with an increase of more than 50%. Influence of the groups led by Brazil and Russia (incl. Syria) increased by about 18% and 12%, respectively; that of the Turkish and Indonesian group by about 11% each; the Indian and Spanish (incl. Mexico and others) groups gained about 10% and 9%. Intended or not, the South African group lost about 8% of its a priori voting power; Saudi Arabia is the greatest loser with roughly 27%. Germany, France and UK each lost between 5% and 7% while voting power of the USA almost stayed constant.

The computations exhibit a simple pattern regarding possible interpretations of 'majority': voting power of the USA is higher for plurality rule than for Copeland than for plurality with runoff; exactly the opposite applies to all other (groups of) countries. Given how controversially reforms of IMF quotas have been debated in the past, it is noteworthy that the influence differences between methods turn out to be bigger than differences between pre and post-reform weights for 20 out of 24

[13]IMF Press Release 16/19, Part 4, holds that "Although the Executive Board may select a Managing Director by a majority of the votes cast, the objective of the Executive Board is to select the Managing Director by consensus . . . ". The same is said in Part 3 about adoption of the "shortlist". Our analysis presumes that a consensus may not exist right away but typically arises in the shadow of straw votes.

[14]Only $\mathcal{PBI}_{\text{Japan}}(r^P|\mathbf{w}_{\text{pre}}) \neq \mathcal{PBI}_{\text{Japan}}(r^P|\mathbf{w}_{\text{post}})$ is not significant. The huge number $6^{24} > 4.7 \cdot 10^{18}$ of profiles $\mathbf{P}$ for 24 players renders exact calculation of $\mathcal{PBI}(\cdot)$ impractical.

	Vote share (%)		$\mathcal{PBI}(r^P\|\mathbf{w})$		$\mathcal{PBI}(r^{PR}\|\mathbf{w})$		$\mathcal{PBI}(r^C\|\mathbf{w})$	
	$\mathbf{w}_{\text{pre}}$	$\mathbf{w}_{\text{post}}$	$\mathbf{w}_{\text{pre}}$	$\mathbf{w}_{\text{post}}$	$\mathbf{w}_{\text{pre}}$	$\mathbf{w}_{\text{post}}$	$\mathbf{w}_{\text{pre}}$	$\mathbf{w}_{\text{post}}$
USA	16.72	16.47	0.7126	0.7030	0.6740	0.6653	0.6880	0.6790
Japan	6.22	6.13	0.1986	0.1989	0.2239	0.2233	0.2164	0.2159
China	3.80	6.07	0.1216	0.1967	0.1404	0.2209	0.1340	0.2135
Netherlands	6.56	5.41	0.2092	0.1755	0.2350	0.1983	0.2277	0.1910
Germany	5.80	5.31	0.1851	0.1720	0.2097	0.1950	0.2024	0.1876
Spain	4.90	5.29	0.1567	0.1718	0.1789	0.1945	0.1717	0.1871
Indonesia	3.93	4.33	0.1254	0.1403	0.1448	0.1607	0.1382	0.1538
Italy	4.22	4.12	0.1349	0.1337	0.1551	0.1533	0.1482	0.1465
France	4.28	4.02	0.1370	0.1306	0.1574	0.1499	0.1507	0.1432
United Kingdom	4.28	4.02	0.1369	0.1304	0.1574	0.1498	0.1506	0.1431
Korea	3.48	3.78	0.1114	0.1226	0.1291	0.1410	0.1230	0.1345
Canada	3.59	3.37	0.1150	0.1093	0.1332	0.1265	0.1268	0.1203
Sweden	3.39	3.28	0.1085	0.1063	0.1259	0.1231	0.1198	0.1171
Turkey	2.91	3.22	0.0932	0.1044	0.1088	0.1209	0.1032	0.1149
South Africa	3.41	3.09	0.1091	0.1001	0.1267	0.1162	0.1205	0.1104
Brazil	2.61	3.06	0.0835	0.0993	0.0979	0.1154	0.0927	0.1096
India	2.80	3.04	0.0898	0.0988	0.1048	0.1147	0.0993	0.1089
Switzerland	2.94	2.88	0.0941	0.0935	0.1097	0.1087	0.1041	0.1030
Russian Federation	2.55	2.83	0.0817	0.0920	0.0957	0.1070	0.0905	0.1015
Iran	2.73	2.54	0.0874	0.0823	0.1024	0.0962	0.0970	0.0910
Utd. Arab Emirates	2.57	2.52	0.0822	0.0817	0.0963	0.0955	0.0911	0.0904
Saudi Arabia	2.80	2.01	0.0896	0.0652	0.1046	0.0767	0.0992	0.0723
Dem. Rep. Congo	1.46	1.62	0.0465	0.0526	0.0555	0.0621	0.0521	0.0584
Argentina	1.84	1.59	0.0587	0.0515	0.0695	0.0610	0.0654	0.0573

Table 5: Influence in IMF Executive Board for pre- and post-reform weights and $m = 3$ (*groups* as of Dec. 2018 indicated by largest member)

Board members. The exceptions are China, Saudi Arabia, and the groups led by Brazil and the Netherlands.

7 General Rule Comparisons

One may wonder if the largest player in a committee – like the USA in the IMF – generally benefits from plurality voting, or if small players typically have greater influence with some form of pairwise choice such as a runoff? And can robust recommendations for maximizing a player's influence be given even if the distribution of voting weights fluctuates over time (like IMF quotas or population-based weights in the EU)? We take a first step beyond single examples and look for possible size biases of our five social choice rules in general. Attention is restricted to small numbers of players and alternatives for a start; namely $n = 3$ and $m = 3$ or 4.

We use the standard projection of the 3-dimensional simplex of relative voting weights to the plane in order to report on rule comparisons for all possible weight distributions among three players: vertices give 100% of voting weight to the indicated player, the midpoint corresponds to $(1/3, 1/3, 1/3)$, etc. Figure 1 and Figures A-1–A-4 in the Appendix present the results of numerically comparing – based on six significant digits – influence of player 1 under some rule ρ vs. rule ρ'. Areas colored green (red) indicate voting weight distributions for which $\mathcal{PBI}_1(N, A, \rho) > (<) \mathcal{PBI}_1(N, A, \rho')$; yellow reflects equality.

7.1 Borda vs. Plurality

The major cases that arise when player 1's influence in Borda vs. plurality committees are compared are numbered in Figure 1.[15] We focus on generic $w_1 \neq w_2 \neq w_3$ and write $\tilde{w}_i = w_i/(w_1 + w_2 + w_3)$, $w^+_{-1} = \max\{\tilde{w}_2, \tilde{w}_3\}$, and $w^-_{-1} = \min\{\tilde{w}_2, \tilde{w}_3\}$. The following recommendations could then be given to an influence-maximizing player 1 if the procedural choice between r^B and r^P is at this player's discretion:

- If you wield the majority of votes (regions 1a,b) impose plurality rule.

  Namely, $\tilde{w}_1 > \frac{2}{3}$ makes you a plurality and Borda dictator (region 1a); $\frac{2}{3} \geq \tilde{w}_1 > \frac{1}{2}$ implies dictatorship only under plurality rule (region 1b).

- Also impose plurality rule (region 5)

[15] Similar distinctions apply to Borda vs. other rules; see Figures A-1 and A-2 (Appendix, p. 33). Darker shades of green or red indicate greater influence differences.

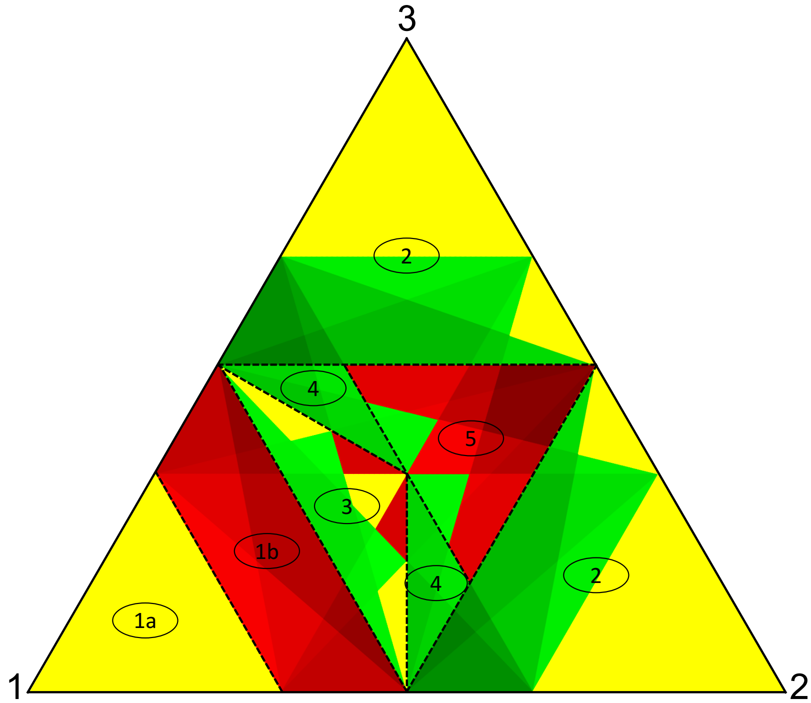

Figure 1: Borda vs. plurality for $m = 3$. Regions colored green (yellow/red) indicate player 1's Borda influence is greater than (equal to/smaller than) plurality influence

- if your weight is smallest and others have a third to half of the votes each ($\frac{1}{3} \leq w_{-1}^{-} < \frac{1}{2}$), or
- if you have less than a third of votes and the largest player falls short of the majority by no more than a quarter of the remaining player's votes ($\frac{1}{2} > w_{-1}^{+} \geq \frac{1}{2} - \frac{1}{4}w_{-1}^{-}$).

- Otherwise (regions 2–4), as a good 'rule of thumb', impose Borda rule instead of plurality. Note that this includes most cases in region 3 where player 1 has a plurality of votes. The only exceptions are two small subregions where all weights are similar but $\tilde{w}_1 > \tilde{w}_{-1}^{+} > \frac{1}{3} > \tilde{w}_{-1}^{-}$.

7.2 Further Pairwise Comparisons

Analogous pairwise influence comparisons are depicted in the Appendix for all weight distributions $\mathbf{w}$ and $r \in \{r^B, r^C, r^{IR}, r^P, r^{PR}\}$. Borda's high responsiveness to weight differences requires detailed case distinctions also in comparison to Copeland and plurality (instant) runoff (Figures A-1 and A-2). When plurality rule is compared

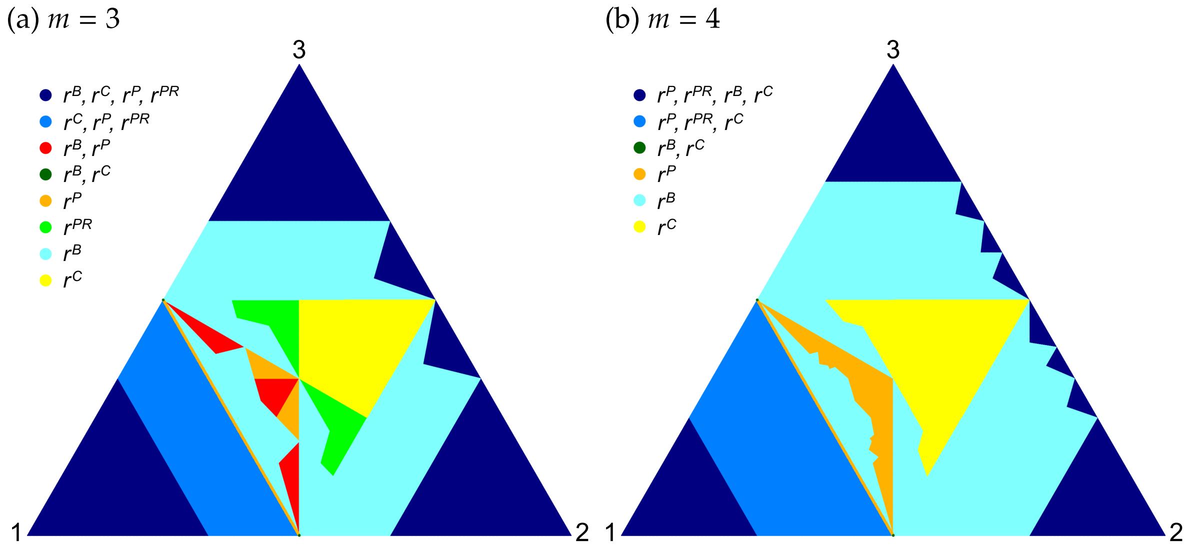

Figure 2: Maximizers of player 1's influence

to either Copeland or plurality (instant) runoff (Figure A-3, p. 34), the recommendation is very intuitive: plurality rule maximizes influence if you have the most votes. Otherwise your influence is greater (at least weakly) under the respective other rule. For Copeland vs. plurality runoff (Figure A-4), the former gives greater influence to you if you have at least the second-most votes.

7.3 Influence Maximization

One can also check directly which of the considered voting rules maximizes a specific player i's a priori voting power $\mathcal{PBI}_i(N, A, r|\mathbf{w})$. Results for any given weight distribution are summarized in Figure 2(a) for $m = 3$ and in Figure 2(b) for $m = 4$.[16] Configurations of same color indicate the same set of influence-maximizing voting rules for player 1. Tongue-in-cheek, Figure 2 provides a map for influence-maximizing chairpersons – or whoever has a say on the adopted voting rule and cares about a specific player's influence. The map might also make subjective impressions that adoption of a specific rule creates bias against players 2 or 3 more objective.

7.4 Transparency

While individual players may seek maximal influence, a constitutional designer more likely cares about aspects such as the *transparency* of a voting arrangement:

[16] Recall $r^{PR} \equiv r^{IR}$ for $n = 3$. Some focal lines or points in the figures have been manually enlarged.

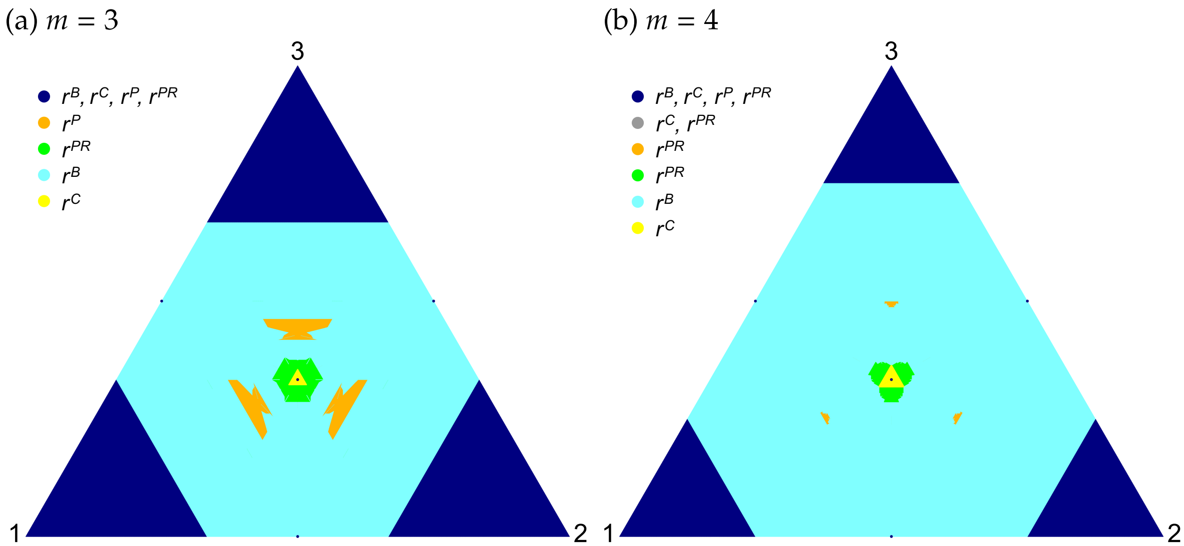

Figure 3: Most transparent voting rules

the induced distribution of power should differ as little as possible from the weight distribution. Weights in supranational decision bodies tend to be agreed on behalf of citizens who are unaware that voting weights and voting power differ. This issue has played a key role in the 'Jagiellonian Compromise' for voting rules of the Council of the EU (e.g., Słomczyński and Życzkowski 2017). A transparent rule avoids paradoxical situations such as Luxembourg casting votes in the Council of Ministers that never mattered for qualified majority decisions in 1958–1972.

One can quantify the misalignment of a given voting weight arrangement and the implied distribution of voting power as $\|\cdot\|_1$-distance

$$d(\mathbf{w}, \mathcal{PBI}(N, A, r|\mathbf{w})) = \sum_{i \in N} \left| \overline{w}_i - \frac{\mathcal{PBI}_i(\cdot)}{\sum_{j \in N} \mathcal{PBI}_j(\cdot)} \right| \tag{25}$$

between relative voting weights $\overline{w}_i := w_i / \sum_{j \in N} w_j$ and power. For instance, relative weights $\overline{\mathbf{w}} = (6, 5, 3)/14$ have a distance of 2.66% to the induced distribution of relative power $(588, 516, 312)/1416$ if Borda rule is used for $m = 3$ (see Section 6.1). The analogous distance is considerably bigger under plurality, plurality runoff, or for pairwise comparisons: 14.29%, 19.21% and 23.81%, respectively.

Figure 3 shows which of the considered voting rules r numerically minimizes $d(\mathbf{w}, \mathcal{PBI}(N, A, r|\mathbf{w}))$ for all weight configurations among three players who decide on $m = 3$ or 4 alternatives.[17] It turns out that Borda rule comes closest to aligning the

[17] Visibility of some areas requires zooming in. In Figure 3(b), for instance, r^C and r^{PR} are both most

distributions of relative weight and power for most configurations. In this sense r^B is the most transparent of the voting rules investigated here.[18]

8 Committees involving other Culture Assumptions

The impartial culture assumption underlying above computations treats preferences over given alternatives $A = \{a_1, \ldots, a_m\}$ as equally probable and independent across players. One may appeal to the 'principle of insufficient reason' for equiprobability, but independence is harder to justify. Committee votes are often preceded by deliberation and discussion. This plausibly induces similarity and dependence between preferences, e.g., over candidates for a position, even when one adopts an a priori perspective that purposely ignores committee members' preference history.

Unfortunately, while there is just one way for preferences to be independent, there are uncountable ways how they may depend on one another – with little guidance which one to presume. As a robustness check and illustration of how preference correlation may affect a player's influence we will consider a family of distributions over $\mathcal{P}(A)^n$ that nests the IC and IAC assumptions: the multivariate version of the urn scheme proposed by Eggenberger and Pólya (1923). It was first used in social choice analysis by Berg (1985).

Berg suggested to conceive of preferences $P_1, \ldots, P_n \in \mathcal{P}(A)$ as formed or revealed in a sequential way akin to iteratively drawing n balls from an urn with balls of $m!$ different colors. Impartiality is reflected by the urn containing each of the $m!$ possible rankings, represented by a ball of specific color, exactly once when the first ranking P_i is drawn for a random player i. The IC model then entails the replacement of the drawn ball by one ball of identical color. This restores the probability for each $\pi \in \mathcal{P}(A)$ to $1/m!$ before a second player's preference ranking is drawn, the respective ball is replaced, and so forth.

Preference dependence can be accommodated by varying the replacement: for fixed $\alpha > 0$, replace any ball drawn by $1 + \alpha$ balls of identical color. The α extra balls that represent the first player's preferences raise the probability of the second

transparent for weights on a thin gray line around the yellow triangle in the middle.

[18] An alternative goal could be to minimize $\|\cdot\|_1$-distance to $(1/3, 1/3, 1/3)$ instead of $\overline{\mathbf{w}}$ in (25). The rules perform equally if (i) $\overline{\mathbf{w}} = (1/3, 1/3, 1/3)$, (ii) $\overline{w}_i = \overline{w}_j = 50\%$ for $i \neq j$, or (iii) $\overline{w}_i > \frac{m-1}{m}$. For $1/2 < \overline{w}_i < \frac{m-1}{m}$, player i is a dictator under r^C, r^P, r^{PR} but not r^B. Then r^B is most *'equalizing'*. r^C is so if $\overline{w}_i < 1/2$ for all i because players then are symmetric under r^C. We also note that random gaps $\overline{w}_i - \mathcal{PBI}_i(\cdot)/\|\mathcal{PBI}(\cdot)\|_1$ that arise for r^C, r^P, r^{PR} if $\overline{\mathbf{w}}$ is drawn uniformly from the simplex are mean-preserving spreads of those for r^B. In that sense r^B also minimizes risk of an individual power misalignment.

or later preference draws to coincide with it, etc. Parameter $\alpha > 0$ thus represents positive preference spillovers or similarity between voters. Analogously $-1 \leq \alpha < 0$ corresponds to negative spillovers and dissimilarity.

The special case of $\alpha = 1$ coincides with the IAC assumption (see Berg 1985): all $\mathbf{n} = (n_1, \ldots, n_{m!}) \in \mathbb{N}_0^{m!}$ with $n_1 + \ldots + n_{m!} = n$ and, for given count $\mathbf{n}$, all preference profiles $\mathbf{P}$ such that $(n_1^{\mathbf{P}}, \ldots, n_{m!}^{\mathbf{P}}) = \mathbf{n}$ are equally probable, where $n_k^{\mathbf{P}}$ denotes the number of players whose preferences coincide with the k-th element of $\mathcal{P}(A)$. In particular, $\alpha = 1$ implies

$$\Pr(\mathbf{P}) = \left[\binom{m! + n - 1}{n} \cdot \binom{n}{n_1^{\mathbf{P}}, \ \ldots, \ n_{m!}^{\mathbf{P}}}\right]^{-1} \tag{26}$$

for any $\mathbf{P} \in \mathcal{P}(A)^n$. Given (26), definitions (8) and (9) specialize to

$$\widehat{\mathcal{SSI}}_i(N, A, \rho) := \sum_{\mathbf{P} \in \mathcal{P}(A)^n} \frac{n_1^{\mathbf{P}}! \cdot \ldots \cdot n_{m!}^{\mathbf{P}}! \cdot (m! - 2)!}{(m! + n - 1)!} \cdot \sum_{P_i' \neq P_i \in \mathcal{P}(A)} \Delta\rho(\mathbf{P}; P_i'), \quad i \in N. \tag{27}$$

with $0! := 1$ and, again normalizing a dictator's influence to 1,

$$\mathcal{SSI}_i(N, A, \rho) := \frac{m! - 1}{m! - (m-1)!} \cdot \widehat{\mathcal{SSI}}_i(N, A, \rho), \quad i \in N. \tag{28}$$

These indices coincide with the classical Shapley-Shubik index when $m = 2$:

Proposition 4. *Let $|A| = 2$. Then*

$$\mathcal{SSI}(N, A, \rho) = \widehat{\mathcal{SSI}}(N, A, \rho) = SSI(N, v_\rho).$$

Proof. Choosing $A = \{0, 1\}$ and using s to denote the number of orderings $1P_j0$ in a given profile $\mathbf{P}$ as well as members of the corresponding coalition $S^{\mathbf{P}}$, we have

$$\begin{aligned}
\mathcal{SSI}_i(N, A, \rho) &= \sum_{\mathbf{P} \in \mathcal{P}(A)^n} \frac{s! \cdot (n-s)!}{(n+1)!} \cdot \sum_{P_i' \neq P_i \in \mathcal{P}(A)} \Delta\rho(\mathbf{P}; P_i') \\
&= \sum_{\substack{\mathbf{P} \in \mathcal{P}(A)^n, \\ 0P_i1}} \frac{s! \cdot (n-s)!}{(n+1)!} [v_\rho(S^{\mathbf{P}} \cup i) - v_\rho(S^{\mathbf{P}})] \\
&\qquad + \sum_{\substack{\mathbf{P} \in \mathcal{P}(A)^n, \\ 1P_i0}} \frac{s! \cdot (n-s)!}{(n+1)!} [v_\rho(S^{\mathbf{P}}) - v_\rho(S^{\mathbf{P}} \setminus i)]
\end{aligned} \tag{29}$$

	$m = 3$	$m = 4$	$m = 5$
$\mathcal{SSI}(r^P\|(6,5,3))$	(0.5714, 0.3929, 0.3929)	(0.7200, 0.3692, 0.3692)	(0.7934, 0.3203, 0.3203)
$\mathcal{SSI}(r^{PR}\|(6,5,3))$	(0.4821, 0.4821, 0.4286)	(0.5631, 0.5631, 0.4800)	(0.5959, 0.5959, 0.4959)
$\mathcal{SSI}(r^B\|(6,5,3))$	(0.6161, 0.5536, 0.3036)	(0.7230, 0.6138, 0.3467)	(0.7604, 0.6441, 0.3808)
$\mathcal{SSI}(r^C\|(6,5,3))$	(0.4732, 0.4732, 0.4732)	(0.5623, 0.5623, 0.5623)	(0.6049, 0.6049, 0.6049)

Table 6: Voting power in committee $(N, A, r|(6,5,3))$ for $|A| = m$ and $r \in \{r^P, r^{PR}, r^{IR}, r^B, r^C\}$ under IAC

$$= \sum_{\substack{\mathbf{P} \in \mathcal{P}(A)^n, \\ 0P_i1}} \left\{ \frac{s! \cdot (n-s)!}{(n+1)!} + \frac{(s+1)! \cdot (n-s-1)!}{(n+1)!} \right\} [v_\rho(S^{\mathbf{P}} \cup i) - v_\rho(S^{\mathbf{P}})]$$

$$= \sum_{\substack{S \subseteq N, \\ i \notin S}} \frac{s! \cdot (n-s-1)!}{n!} [v_\rho(S \cup i) - v_\rho(S)] = SSI(N, v_\rho)$$

■

In the toy example investigated in Section 6.1, profile $\mathbf{P} = (bca, abc, cba)$ corresponds to $\mathbf{n} = (1,0,0,1,0,1)$ with $\mathcal{P}(A) = \{abc, acb, bac, bca, cab, cba\}$. So the perturbations that we identified in Table 2 as swinging the Borda winner contribute only $(4/1344, 3/1344, 0)$ to $\mathcal{SSI}(\cdot)$. This is less than under the IC assumption ($\alpha = 0$) because the preference affiliation reflected by $\alpha = 1$ reduces the probability of profiles where all players have distinct preferences. Relatively few Borda swings arise for the now more likely profiles where all players have identical preferences. Overall we obtain dictator-normalized influences of

$$\mathcal{SSI}(N, A, r^B|(6,5,3)) \approx (0.6161, 0.5536, 03036) \tag{30}$$

for $m = 3$ under IAC, compared to $\mathcal{PBI}(\cdot) \approx (0.6806, 0.5972, 0.3611)$ in the IC case.

The intuition that preference similarities under IAC tend to make swing opportunities less likely is broadly confirmed by comparing influence figures reported in Table 4 to Table 6. But $\mathcal{SSI}_i(\cdot) > \mathcal{PBI}_i(\cdot)$ is possible: the influence of players 2 and 3 for $m = 5$ and plurality rule r^P is slightly larger under IAC than IC. The explanation is that outcomes are sensitive to player 2 and 3's preferences under r^P if and only if both have an identical top preference that differs from player 1's. Similarity between players, as captured by $\alpha > 0$, shifts probability towards events where two or all of them have identical preferences. That players 2 and 3 agree but player 1 does not is made more likely for a range of values that includes $\alpha = 1$.

(a) Borda rule r^B (b) Plurality rule r^P

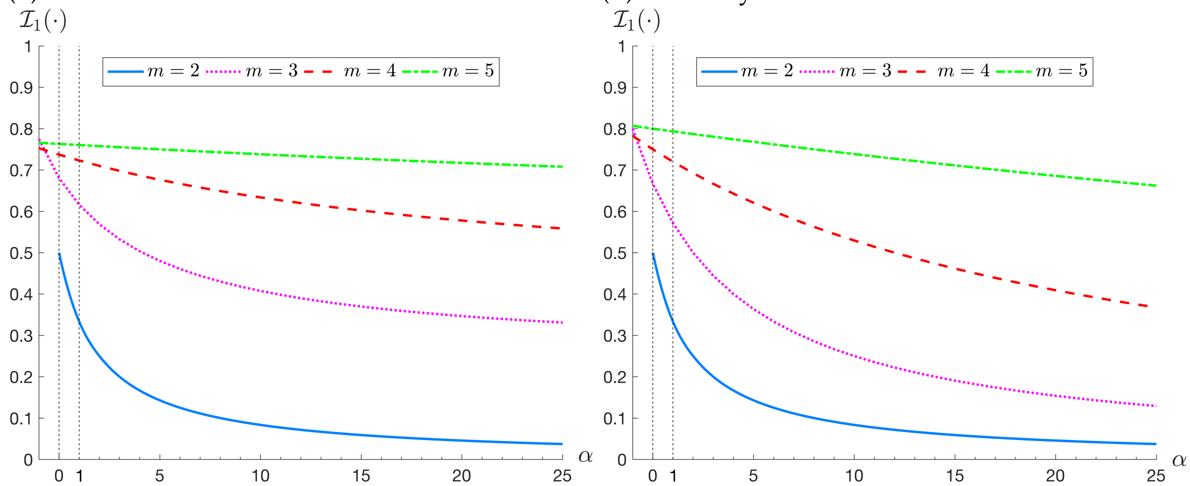

Figure 4: Player 1's influence $\mathcal{I}_1(N, A, r|(6,5,3))$ for $|A| = m$ and Pólya-Eggenberger model with preference spillover $\alpha \in \{-1, 0, 1, \ldots, 25\}$ (interpolated)

As α gets large, all players perfectly agree for more and more preference draws. For the considered committees this ultimately lowers individual influence of all players, at rates that depend on the number m of alternatives. Figure 4 illustrates this for player 1 and $\alpha \in \{-1, 0, 1, \ldots, 25\}$ when $r \in \{r^B, r^P\}$ and $m \in \{2, 3, 4, 5\}$.

Figures on influence in the IMF Executive Board for $\alpha = 1$ are reported in the Appendix (Table A-1). As could already be suspected based on IMF applications of the classical PBI and SSI, individual influence is considerably smaller under IAC than IC, as positive spillovers make profiles less likely in which a single preference perturbation can change the outcome. The key conclusions obtained for the IC assumption are robust, however: influence differences between voting methods are larger than those between pre- and post-reform weights for most members and plurality runoff rule gives greatest influence to all members, except the USA. The main qualitative difference between results for IC and IAC is that voting power of the USA is highest for Copeland rule under the latter (rather than plurality).

One can also compare the considered voting procedures under IAC with the objective of maximizing player 1's influence or transparency (see Figures A-5 and A-6 in the Appendix). Some regions of relative weights can be identified in which, e.g., r^{PR} maximizes 1's influence under IC while r^B or r^C do under IAC (compare the green regions in Figures 2 vs. A-5); or in which r^P is most transparent under IC while r^B is under IAC (see orange regions in Figures 3(a) vs. A-6(a)). Overall, the respective findings obtained under the IC assumption are however surprisingly robust to the positive preference spillovers reflected by IAC.

9 Concluding Remarks

The take-home message of our investigation is that the choice of a voting rule matters not just for particular preference constellations: it drives the a priori balance of power in a committee in a quantifiable way. Voting weights and aggregation method determine together how much collective outcomes respond – on average – to the wishes of individual decision makers. Traditional measures of a priori voting power, such as the Penrose-Banzhaf or Shapley-Shubik indices, fail to capture this and may report spurious symmetries for $m > 2$ alternatives. We propose generalizations that fill this gap.

The indices suggested in this paper are proportional to the probability for a random individual preference perturbation to affect the outcome. A null player's voting power is automatically zero; a dictator player's power can easily be normalized to one. How method and weights jointly distribute power has been illustrated for generic committees involving three voter groups and the election of the IMF's Managing Director. Similar analysis could illuminate the distribution of voting power on corporate boards, party conventions, multi-candidate primaries, and so on.

Evaluating the effects of adopting a voting method already at an ex ante stage complements case studies that document how choice of a particular method may have affected specific big political decisions. See, e.g., Leininger (1993) on making Berlin vs. Bonn the capital of Germany after reunification; Tabarrok and Spector (1999) on electoral causes of the US civil war; or Maskin and Sen (2016) on US primary elections and the nomination of Trump.

For decisions on three or four options, we have identified and compared the power implications of five prominent rules for all conceivable weight distributions among three players. It turns out that relatively coarse information about the distribution of weights suffices for ranking the influence of a large, middle, or small player on voting outcomes across rules, or for determining how well weights and influence are aligned across players. We leave it to future research to further generalize the probability assumptions investigated here and to add a variety of other voting methods to the picture (cf. Nurmi 2006, Ch. 7, or Laslier 2012). The analysis might also be extended to multi-winner elections (see, e.g., Elkind et al. 2017) or to strategic voting in restricted preference domains.

References

Aleskerov, F., V. Kalyagin, and K. Pogorelskiy (2008). Actual voting power of the IMF members based on their political-economic integration. *Mathematical and Computer Modelling 48*(9–10), 1554–1569.

Aleskerov, F. and E. Kurbanov (1999). Degree of manipulability of social choice procedures. In A. Alkan, C. D. Aliprantis, and N. C. Yannelis (Eds.), *Current Trends in Economics*, pp. 13–27. Berlin: Springer.

Alonso-Meijide, J. M. and C. Bowles (2005). Generating functions for coalitional power indices: an application to the IMF. *Annals of Operations Research 137*(1), 21–44.

Amer, R., F. Carreras, and A. Magãna (1998). Extension of values to games with multiple alternatives. *Annals of Operations Research 84*(0), 63–78.

Aumann, R. (1987). Game theory. In J. Eatwell, M. Milgate, and P. Newman (Eds.), *The New Palgrave: A Dictionary of Economics*, pp. 460–482. London: MacMillan.

Banzhaf, J. F. (1965). Weighted voting doesn't work: a mathematical analysis. *Rutgers Law Review 19*(2), 317–343.

Berg, S. (1985). Paradox of voting under an urn model: the effect of homogeneity. *Public Choice 47*, 377–387.

Bolger, E. M. (1986). Power indices for multicandidate voting games. *International Journal of Game Theory 15*(3), 175–186.

Bouton, L. (2013). A theory of strategic voting in runoff elections. *American Economic Review 103*(4), 1248–1288.

Chua, V. C. H., C. H. Ueng, and H. C. Huang (2002). A method for evaluating the behavior of power indices in weighted plurality games. *Social Choice and Welfare 19*(3), 665–680.

Dragan, I. (1996). New mathematical properties of the Banzhaf value. *European Journal of Operational Research 95*(2), 451–463.

Dubey, P. (1975). On the uniqueness of the Shapley value. *International Journal of Game Theory 4*(3), 131–139.

Dubey, P. and L. Shapley (1979). Mathematical properties of the Banzhaf power index. *Mathematics of Operations Research 4*(2), 99–131.

Eggenberger, F. and G. Pólya (1923). Über die Statistik verketteter Vorgänge. *Zeitschrift füer Angewandte Mathematik und Mechanik 3*, 279–289.

Elkind, E., P. Faliszewski, P. Skowron, and A. Slinko (2017). Properties of multiwinner voting rules. *Social Choice and Welfare 48*(3), 599–632.

Felsenthal, D. S. and M. Machover (1997). Ternary voting games. *International Journal of Game Theory 26*(3), 335–351.

Felsenthal, D. S. and M. Machover (1998). *The Measurement of Voting Power*. Cheltenham, UK: Edward Elgar.

Freixas, J. (2005a). Banzhaf measures for games with several levels of approval in the input and output. *Annals of Operations Research 137*(1), 45–66.

Freixas, J. (2005b). The Shapley-Shubik index for games with several levels of approval in the input and output. *Decision Support Systems 39*(2), 185–195.

Freixas, J. and W. S. Zwicker (2003). Weighted voting, abstention, and multiple levels of approval. *Social Choice and Welfare 21*(3), 399–431.

Freixas, J. and W. S. Zwicker (2009). Anonymous yes-no voting with abstention and multiple levels of approval. *Games and Economic Behavior 67*(2), 428–444.

Gibbard, A. (1973). Manipulation of voting schemes: a general result. *Econometrica 41*(4), 587–601.

Holler, M. J. and H. Nurmi (Eds.) (2013). *Power, Voting, and Voting Power: 30 Years After*. Heidelberg: Springer.

Hsiao, C.-R. and T. E. S. Raghavan (1993). Shapley value for multichoice cooperative games, I. *Games and Economic Behavior 5*(2), 240–256.

Kelly, J. S. (1993). Almost all social choice rules are highly manipulable, but a few aren't. *Social Choice and Welfare 10*(2), 161–175.

Kurz, S., A. Mayer, and S. Napel (2020). Weighted committee games. *European Journal of Operational Research 282*(3), 972–979.

Laruelle, A. and F. Valenciano (2008). *Voting and Collective Decision-Making*. Cambridge: Cambridge University Press.

Laruelle, A. and F. Valenciano (2012). Quaternary dichotomous voting rules. *Social Choice and Welfare 38*(3), 431–454.

Laslier, J.-F. (2012). And the loser is ... plurality voting. In D. S. Felsenthal and M. Machover (Eds.), *Electoral Systems: Paradoxes, Assumptions, and Procedures*, pp. 327–351. Berlin: Springer.

Leech, D. (2003). Computing power indices for large voting games. *Management Science 49*(6), 831–837.

Leech, D. and R. Leech (2013). A new analysis of a priori voting power in the IMF: Recent quota reforms give little cause for celebration. In M. J. Holler and H. Nurmi (Eds.), *Power, Voting, and Voting Power: 30 Years After*, pp. 389–410. Heidelberg: Springer.

Leininger, W. (1993). The fatal vote: Berlin versus Bonn. *Finanzarchiv 50*(1), 1–20.

Maskin, E. and A. Sen (2016). How majority rule might have stopped Donald Trump. *New York Times*. April 28, 2016.

Myerson, R. B. (1999). Theoretical comparisons of electoral systems. *European Economic Review 43*(4-6), 671–697.

Napel, S. (2019). Voting power. In R. Congleton, B. Grofman, and S. Voigt (Eds.), *Oxford Handbook of Public Choice*, Volume 1, Chapter 6, pp. 103–126. Oxford: Oxford University Press.

Napel, S. and M. Widgrén (2004). Power measurement as sensitivity analysis: a unified approach. *Journal of Theoretical Politics 16*(4), 517–538.

Nitzan, S. (1985). The vulnerability of point-voting schemes to preference variation and strategic manipulation. *Public Choice 47*(2), 349–370.

Nurmi, H. (2006). *Models of Political Economy*. London: Routledge.

Ortmann, K. M. (1998). Conservation of energy in value theory. *Mathematical Methods of Operations Research 47*(3), 423–449.

Parker, C. (2012). The influence relation for ternary voting games. *Games and Economic Behavior 75*(2), 867–881.

Penrose, L. S. (1946). The elementary statistics of majority voting. *Journal of the Royal Statistical Society 109*(1), 53–57.

Regenwetter, M., B. Grofman, A. A. J. Marley, and I. M. Tsetlin (2012). *Behavioral Social Choice* (2nd edition ed.). New York, NY: Cambridge University Press.

Riker, W. H. (1986). The first power index. *Social Choice and Welfare 3*(4), 293–295.

Satterthwaite, M. A. (1975). Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare function. *Journal of Economic Theory 10*(2), 187–217.

Shapley, L. S. (1953). A value for *n*-person games. In H. W. Kuhn and A. W. Tucker (Eds.), *Contributions to the Theory of Games*, Volume II, pp. 307–317. Princeton, NJ: Princeton University Press.

Shapley, L. S. and M. Shubik (1954). A method for evaluating the distribution of power in a committee system. *American Political Science Review 48*(3), 787–792.

Słomczyński, W. and K. Życzkowski (2017). Degressive proportionality in the European Union. In European Parliament (Policy Department for Citizens Rights and Constitutional Affairs) (Ed.), *The Composition of the European Parliament*. Avalaible at `www.europarl.europa.eu/RegData/etudes/IDAN/2017/583117/IPOL_IDA(2017)583117_EN.pdf`.

Smith, D. A. (1999). Manipulability measures of common social choice functions. *Social Choice and Welfare 16*(4), 639–661.

Tabarrok, A. and L. Spector (1999). Would the Borda Count have avoided the civil war? *Journal of Theoretical Politics 11*(2), 261–288.

Tchantcho, B., L. D. Lambo, R. Pongou, and B. M. Engoulou (2008). Voters' power in voting games with abstention: influence relation and ordinal equivalence of power theories. *Games and Economic Behavior 64*(1), 335–350.

Von Neumann, J. and O. Morgenstern (1953). *Theory of Games and Economic Behavior* (3rd ed.). Princeton, NJ: Princeton University Press.

– Appendix –

Further Comparisons of Voting Rules for $m = 3$

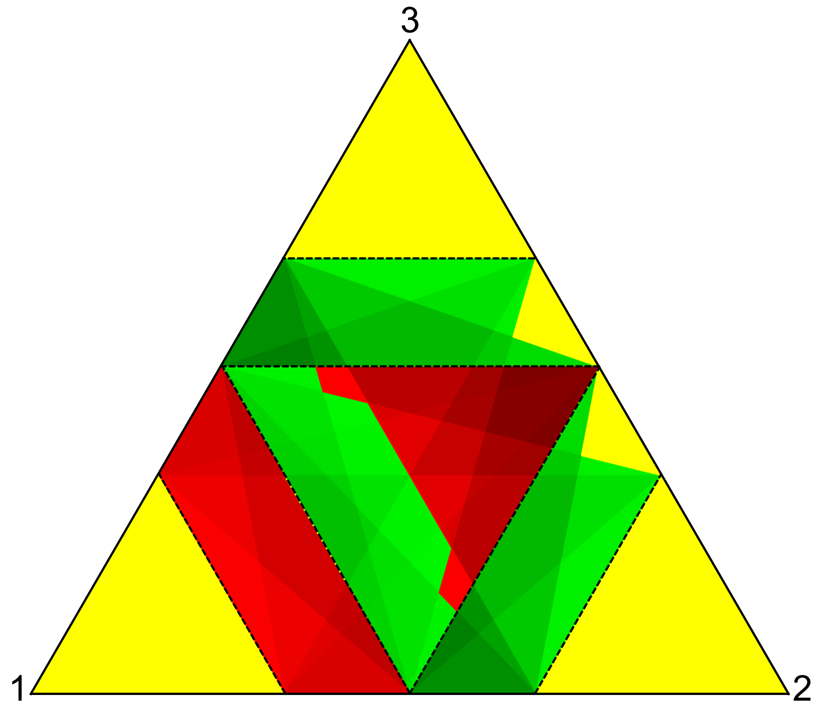

Figure A-1: Borda vs. Copeland

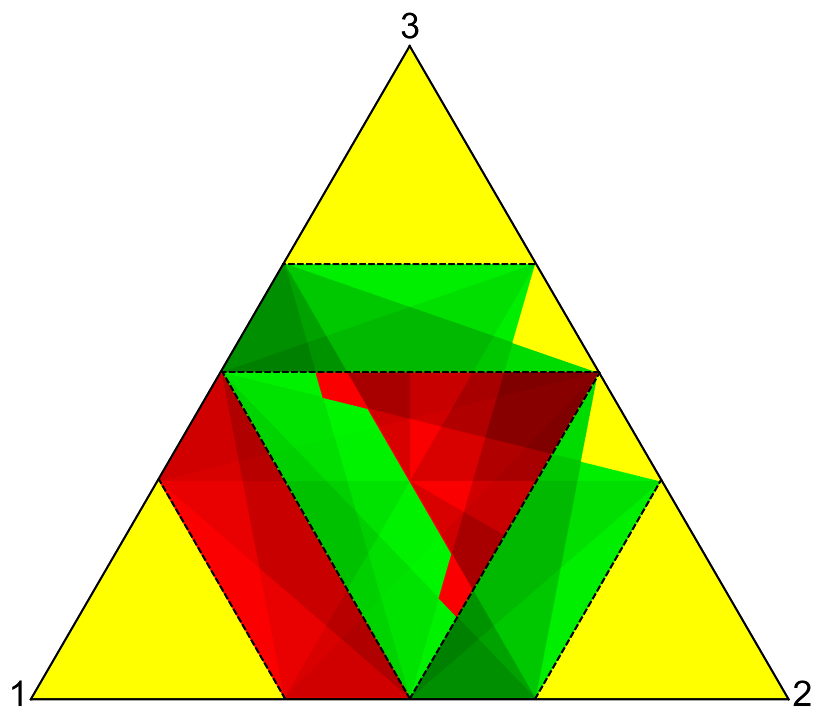

Figure A-2: Borda vs. plurality (instant) runoff

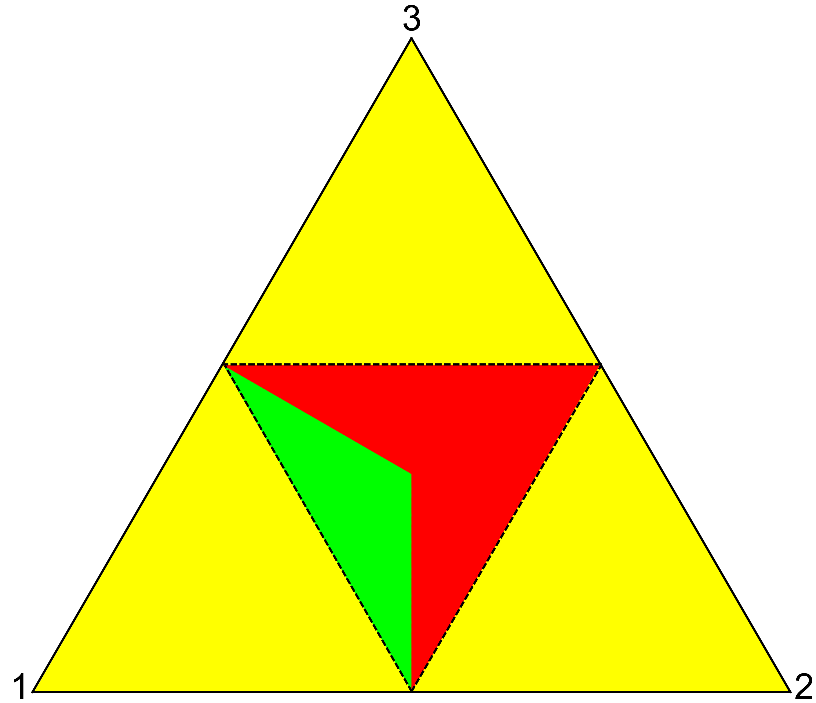

Figure A-3: Plurality vs. plurality (instant) runoff and Copeland

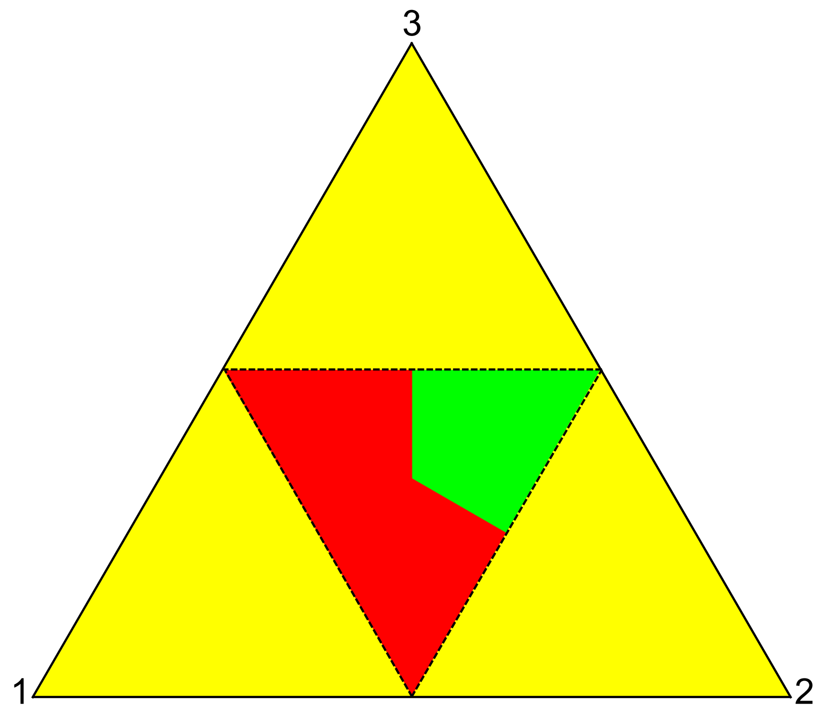

Figure A-4: Copeland vs. plurality (instant) runoff

Influence in IMF Executive Board under IAC

	$\mathcal{SSI}(r^P\|\mathbf{w})$		$\mathcal{SSI}(r^{PR}\|\mathbf{w})$		$\mathcal{SSI}(r^C\|\mathbf{w})$	
	$\mathbf{w}_{\text{pre}}$	$\mathbf{w}_{\text{post}}$	$\mathbf{w}_{\text{pre}}$	$\mathbf{w}_{\text{post}}$	$\mathbf{w}_{\text{pre}}$	$\mathbf{w}_{\text{post}}$
USA	0.3583	0.3516	0.3683	0.3619	0.3716	0.3649
Japan	0.1139	0.1124	0.1261	0.1245	0.1240	0.1223
China	0.0684	0.1110	0.0769	0.1230	0.0751	0.1209
Netherlands	0.1203	0.0986	0.1329	0.1097	0.1308	0.1076
Germany	0.1057	0.0964	0.1175	0.1075	0.1153	0.1054
Spain	0.0888	0.0963	0.0992	0.1072	0.0971	0.1049
Indonesia	0.0707	0.0784	0.0795	0.0878	0.0776	0.0858
Italy	0.0761	0.0745	0.0853	0.0834	0.0834	0.0815
France	0.0773	0.0725	0.0867	0.0814	0.0847	0.0795
United Kingdom	0.0772	0.0725	0.0867	0.0814	0.0847	0.0795
Korea	0.0625	0.0681	0.0703	0.0766	0.0686	0.0747
Canada	0.0646	0.0605	0.0727	0.0682	0.0709	0.0665
Sweden	0.0608	0.0588	0.0686	0.0664	0.0668	0.0647
Turkey	0.0521	0.0578	0.0589	0.0651	0.0573	0.0634
South Africa	0.0612	0.0553	0.0689	0.0625	0.0671	0.0608
Brazil	0.0466	0.0549	0.0529	0.0621	0.0514	0.0604
India	0.0501	0.0546	0.0567	0.0617	0.0552	0.0600
Switzerland	0.0525	0.0516	0.0594	0.0583	0.0578	0.0567
Russian Federation	0.0456	0.0507	0.0517	0.0574	0.0503	0.0558
Iran	0.0489	0.0453	0.0553	0.0514	0.0538	0.0499
Utd. Arab Emirates	0.0458	0.0450	0.0519	0.0511	0.0505	0.0496
Saudi Arabia	0.0500	0.0358	0.0567	0.0407	0.0551	0.0395
Dem. Rep. Congo	0.0258	0.0288	0.0295	0.0329	0.0286	0.0318
Argentina	0.0326	0.0282	0.0372	0.0322	0.0360	0.0311

Table A-1: Influence in IMF Executive Board for pre- and post-reform weights and $m = 3$ under IAC (*groups* as of Dec. 2018 indicated by largest member)

Influence and Transparency Maximizers under IAC

(a) $m = 3$ (b) $m = 4$

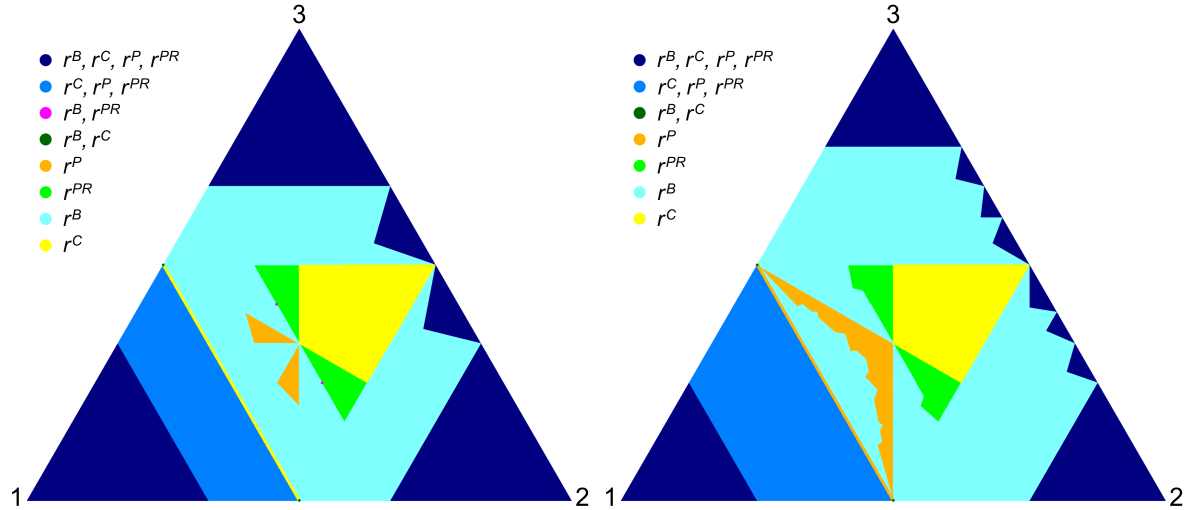

Figure A-5: Maximizers of player 1′s influence under IAC

(a) $m = 3$ (b) $m = 4$

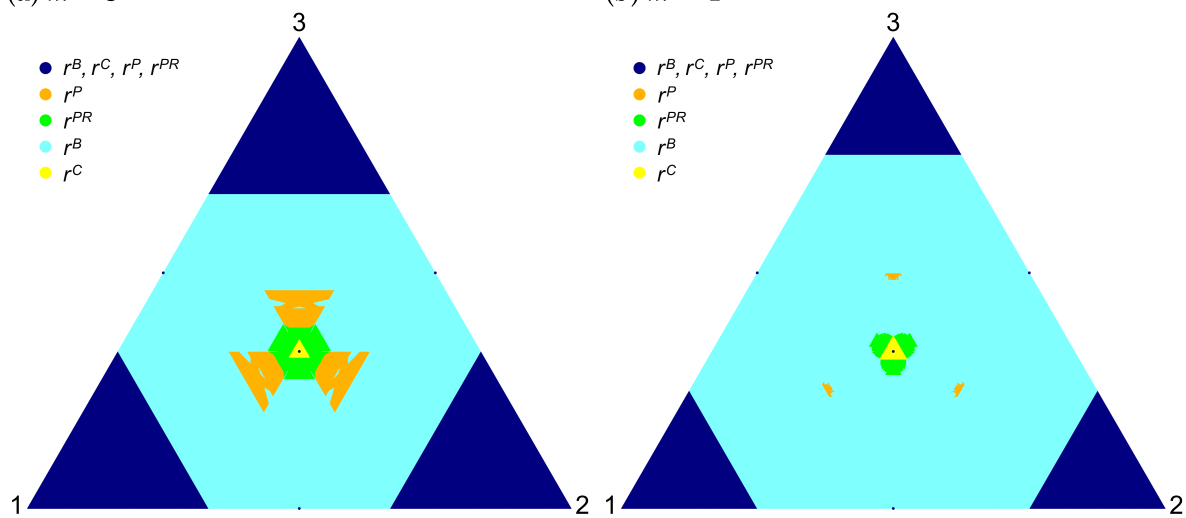

Figure A-6: Most transparent voting rules under IAC