Weighted Committee Games

Sascha Kurz
Dept. of Mathematics, University of Bayreuth, Germany
sascha.kurz@uni-bayreuth.de

Alexander Mayer
Dept. of Economics, University of Bayreuth, Germany
alexander.mayer@uni-bayreuth.de

Stefan Napel
Dept. of Economics, University of Bayreuth, Germany
stefan.napel@uni-bayreuth.de

August 28, 2018

Abstract

Players in a committee, council, or electoral college often wield asymmetric numbers of votes. Binary decision environments are then conventionally modeled as *weighted voting games*. We introduce *weighted committee games* in order to describe decisions on three or more alternatives in similarly succinct fashion. We compare different voting weight configurations for plurality, Borda, Copeland, and antiplurality rule. We identify the respective number and geometry of structurally non-equivalent committees. They have escaped notice so far and determine voting equilibria, the distribution of power, and other aspects of collective choice.

Keywords: voting games · weighted voting · geometry of voting · voting power · Borda rule · Copeland rule · plurality · antiplurality

JEL codes: C71 · D71 · C63

We are indebted to Hannu Nurmi for drawing our attention to a lacuna that this work begins to fill. We are also grateful to him, Steven Brams, Dan Felsenthal, Bernard Grofman, Nicola Maaser, Joel Sobel, and William Zwicker for feedback on earlier drafts, as well as to seminar and conference audiences in Bamberg, Bayreuth, Berlin, Bremen, Dagstuhl, Delmenhorst, Graz, Hagen, Hamburg, Hanover, Leipzig, Moscow, Munich, Rome, Seoul, and Turku.

1 Introduction

Consider a committee, council, corporate board, etc. that involves three players (parties, groups, shareholders, delegations). Suppose the first wields 6 votes, the second 5 votes, and the third 2 votes. Will their collective choices differ, ceteris paribus, from those resulting if each player wielded 5 votes? Or, say, from outcomes for a $(48\%, 24\%, 28\%)$ distribution of votes?

The way in which voting weights translate into collective decisions and how they affect the influence of the respective decision makers are old concerns for institutional design. See Riker (1986), for instance, on reactions by delegate Luther Martin from Maryland to the Constitutional Convention in Philadelphia in 1787. Notwithstanding residual disagreement on which measures of power or success are the most meaningful in a given context, the structural properties of weighted voting are today well understood for decisions on two alternatives. It is easy to see, e.g., that for all of the above weight distributions any pair of players can jointly implement their preferred alternative. The two form a winning coalition irrespective of whether they have an $11:2$, $7:6$, $10:5$, or $52\%:48\%$ majority. As long as each player wields positive but less than half the total weight, *all* distributions of votes among three players are equivalent for a simple majority requirement. They amount to different *weighted representations* of the same mathematical structure known as a *simple voting game*. The literature has formalized numerous related results.

But what if the committee is to choose from three or more candidates? Very little is known then. Consider the simplest case: the committee uses plurality rule and always selects the candidate who received the most votes. Now player 1 has greater say for weights of $(6, 5, 2)$ than for equal ones. Namely, whenever players 2 and 3 fail to agree, player 1 is decisive and his or her favorite candidate wins with a tally of $6:5:2$, $11:2$, or $8:5$. The same plurality winners would result for $(48\%, 24\%, 28\%)$, i.e., committees with voting weights of $(6, 5, 2)$ and $(48\%, 24\%, 28\%)$ are structurally equivalent – but one with $(5, 5, 5)$ is not. One can conceive of the former as different weighted representations of the same *committee game*, referring to the combination of a set of n players, a set of m alternatives, and a particular mapping from n-tuples of preferences to a winning alternative.

The goal of this paper is to extend the knowledge on equivalent weighted voting environments from two to more alternatives. We study four standard aggregation methods: plurality, Borda, Copeland, and antiplurality rule. These can produce four different winners for the same profile of preferences. We show that the methods also differ widely in the degree to which voting weights matter. We establish, for instance, that there exist only 4 structurally different Copeland committees but 51 Borda committees when $n = m = 3$. Our findings do not depend on whether sincere preference statements or strategic votes are considered.

Committees that decide between two alternatives have received wide attention. Von Neumann and Morgenstern (1953) started their formal analysis by introducing simple voting games. Shapley and Shubik (1954) and Banzhaf (1965) constructed corresponding indices of voting power. Their applications range from the US Electoral College, UN Security Council, and EU Council to governing bodies of the IMF and private corporations. See Mann and Shapley (1962), Riker and Shapley (1968), Owen (1975), or Brams (1978) for seminal contributions. They and more recently Barberà and Jackson (2006), Felsenthal and Machover (2013), Koriyama et al. (2013), Kurz et al. (2017), and many others have sought to quantify the links between voting weights and collective choices to evaluate democratic playing fields from a fairness or welfare perspective.

The extent to which different voting weights make a real or only a superficial difference has practical relevance. Weighted committee games offer the potential to extend the respective analysis to decision bodies that face general non-binary options. For example, voting rights among the 24 Directors of the International Monetary Fund's Executive Board were reformed in 2016. Is there a possibility that this will affect any decisions, such as its choice of the next IMF Managing Director? The Executive Board declared (IMF Press Release 2016/19) that in the future a winner from a shortlist of at most three candidates shall be adopted "by a majority of the votes cast". Suppose this means (i) receiving the most votes (plurality rule). Have changes of the distribution of IMF drawing rights, hence votes, then made a difference? And would it make a difference to interpret the declaration as calling instead for (ii) a two-candidate runoff if nobody gets an outright majority (plurality runoff rule) or (iii) securing the highest number of pairwise majority wins against competitors (Copeland rule)? Both types of questions – comparing distinct vote distributions for a given rule or different rules for a given distribution – are about equivalences between committees that we formalize and study in this paper.

We are not concerned with any particular voting body here but develop a parsimonious framework for classifying non-binary voting structures. Different compositions of committees – monotonically related to an underlying scale such as population, shareholdings, etc. or not – are investigated concerning their possible equivalence under a given basic voting method. We seek to identify all structurally distinct weight distributions to help assessing, for instance, if nominal differences in political representation translate into real ones. We determine minimal representations for all pertinent committee games for selected n and m. Comprehensive lists of games only existed for $m = 2$ alternatives so far; we include analogous results for $m \geq 3$ for antiplurality, Borda, Copeland, and plurality rule. These extensions could be applied, e.g., to establish sharp bounds on the numbers of voters and alternatives that permit certain monotonicity violations or paradoxes (cf. Felsenthal and Nurmi 2017); to generalize rule-specific findings on manipulability from one to infinitely many equivalent committees (see Aleskerov and Kurbanov 1999); or to check robustness of voting equilibria to small reallocations of voting weights (cf. Myerson and Weber 1993, Bouton 2013, or Buenrostro et al. 2013). The number of (binary) weighted voting games with a 50%-majority requirement used to be known only for up to $n = 6$ voter groups; we provide it for $n = 7$, 8, and 9. We also give a glimpse of the alluring geometry of weighted committee games.

2 Related concepts

Our analysis concerns arbitrary mappings from n-tuples of strict preferences over m alternatives to a winning alternative. We seek to connect a given mapping to an anonymous baseline decision rule in the same way as weighted representations of a simple voting game described by player set N and coalitional function v connect it to simple majority or supermajority rule.

Simple voting games and the subclass of weighted voting games (i.e., those that have weighted representations) received a complete chapter's attention by von Neumann and Morgenstern (1953, Ch. 10). Taylor and Zwicker (1999) devoted a full-length monograph to them and investigations continue. See, e.g., Kurz and Tautenhahn (2013) on open challenges in classifying and enumerating simple voting games in the tradition of Shapley (1962). Machover and Terrington (2014) investigate simple voting games as mathematical objects in their own right and connect their algebraic structure to seemingly unrelated areas of mathematics. Houy and Zwicker (2014) or Freixas et al. (2017) document ongoing progress on the problem of verifying if a given game (N, v) is weighted.

Notwithstanding such advancements the literature has increasingly acknowledged that the presumption of dichotomous decision making is a limitation. Many committee decisions allow more than two outcomes. Even for binary motions, voters usually can abstain, stay away from the ballot, express different intensities of support, etc.

This has led to generalizations of simple voting games to multiple levels of approval. For instance, Felsenthal and Machover (1997), Tchantcho et al. (2008) and Parker (2012) have considered *ternary voting games* with the option to support a proposal, to abstain, or to reject it. *Quaternary voting games* introduced by Laruelle and Valenciano (2012) add the possibility not to participate in a ballot. The case of an arbitrary finite number of individual actions translating into one of finitely many collective outcomes has been addressed by Hsiao and Raghavan (1993) and Freixas and Zwicker (2003, 2009). In their *(j, k)-games* each player expresses one of j linearly ordered levels of approval and every resulting j-partition of player set N is mapped to one of k ordered output levels.

Committees that determine quantities like grades, interest rates, budget sizes, etc. can be modeled as (j, k)-games. But the assumption of ordered actions and outcomes is problematic for alternatives with multidimensional attributes – for instance, if the committee is to select from several policy options, locations of a facility, job candidates, etc. Pertinent extensions of simple voting games have been introduced as *multicandidate voting games* by Bolger (1986) and taken up as *simple r-games* by Amer et al. (1998). These are the most closely related concepts in the literature to weighted committee games as far as we are aware. In particular, weighted plurality committees (as defined below) have featured in the framework of Bolger and Amer et al. as "simple plurality games" and "relative majority r-games". However, the respective analysis was restricted to variations of plurality voting; it concerned values and power indices rather than structural investigation of the underlying games. We seem the first to find, e.g., that there are no more than 36 distinct "simple plurality games" with four players[1] and so only 36 different distributions of power can arise.

3 Notation and definitions

3.1 Committees and simple voting games

We consider finite sets N of $n \geq 1$ players or voters such that each voter $i \in N$ has strict preferences P_i over the set $A = \{a_1, \ldots, a_m\}$ of $m \geq 2$ alternatives. $\mathcal{P}(A)$ denotes the set of all $m!$ strict preference orderings on A. A *(resolute) social choice rule* $\rho\colon \mathcal{P}(A)^n \to A$

[1]Chua et al. (2002) identified the eight games that cannot generate ties for $m = 3$.

maps each profile $\mathbf{P} = (P_1, \ldots, P_n)$ to a single winning alternative $a^* = \rho(\mathbf{P})$. The combination (N, A, ρ) of a set of voters, a set of alternatives and a particular social choice rule will be referred to as a *committee game* or just as a *committee*.

For given N and A, there are $m^{(m!^n)}$ distinct rules ρ. Those that treat all voters $i \in N$ symmetrically will play a special role in our analysis: suppose preference profile $\mathbf{P}'$ results from applying a permutation $\pi\colon N \to N$ to profile $\mathbf{P}$, so $\mathbf{P}' = (P_{\pi(1)}, \ldots, P_{\pi(n)})$. Then ρ is *anonymous* if for all such $\mathbf{P}$, $\mathbf{P}'$ the winning alternative $a^* = \rho(\mathbf{P}) = \rho(\mathbf{P}')$ is the same. We will write r instead of ρ if we want to highlight that a considered rule is anonymous, i.e., we impose no restrictions on general social choice rules denoted by ρ but require anonymity for rules denoted by $r\colon \mathcal{P}(A)^n \to A$.

For $m = 2$ and binary alternatives $a_1 = 1$ and $a_2 = 0$, it is common to describe ρ by a *coalitional function* $v\colon 2^N \to \{0, 1\}$ with $v(S) = 1$ when $1\, P_i\, 0$ for all $i \in S$ implies $\rho(\mathbf{P}) = 1$. Sets $S \subseteq N$ with $v(S) = 1$ are called *winning coalitions*. The pair (N, v) is referred to as a *simple voting game*.

A simple voting game (N, v) is *weighted* and also called *weighted voting game* if there exists a non-negative vector $\mathbf{w} = (w_1, \ldots, w_n)$ of weights and a positive quota q such that $v(S) = 1$ if and only if $\sum_{i \in S} w_i \geq q$. One then refers to pair $(q; \mathbf{w})$ as a *(weighted) representation* of (N, v) and denotes the respective game by $[q; \mathbf{w}]$, i.e., $(N, v) = [q; \mathbf{w}]$. It is without loss of generality to focus on integer numbers: given $q \in \mathbb{R}_{++}$ and $\mathbf{w} \in \mathbb{R}^n_+$ one can always find $q' \in \mathbb{N}$ and $\mathbf{w}' \in \mathbb{N}^n_0$ such that $[q; \mathbf{w}] = [q'; \mathbf{w}']$.

Somewhat involved analogues of winning coalitions and coalitional functions exist for $m > 2$. For instance, Moulin (1981) introduced *veto functions* to succinctly describe the outcomes that given coalitions of players could prevent if they coordinated. Different types of *effectivity functions* clarify the power structure associated with a rule ρ by enumerating the sets of alternatives that given coalitions of voters can force $\rho(\mathbf{P})$ to lie in. See Peleg (1984). We provide a different perspective by investigating analogues to weightedness of a simple voting game for the domain of general committee games.

3.2 Four anonymous social choice rules

We will define weightedness of general rules ρ relative to some fixed anonymous rule r. For the latter we here focus on four standard social choice rules with lexicographic tie breaking. Their definitions are summarized in Table 1.

Under *plurality rule* r^P each voter names his or her top-ranked alternative and

Rule	Winning alternative at preference profile $\mathbf{P}$
Anti-plurality	$r^A(\mathbf{P}) \in \arg\min_{a \in A} \left\| \{i \in N \mid \forall a' \neq a \in A \colon a' P_i a\} \right\|$
Borda	$r^B(\mathbf{P}) \in \arg\max_{a \in A} \sum_{i \in N} b_i(a, \mathbf{P})$
Copeland	$r^C(\mathbf{P}) \in \arg\max_{a \in A} \left\| \{a' \in A \mid a \succ_M^{\mathbf{P}} a'\} \right\|$
Plurality	$r^P(\mathbf{P}) \in \arg\max_{a \in A} \left\| \{i \in N \mid \forall a' \neq a \in A \colon a P_i a'\} \right\|$

Table 1: Considered anonymous social choice rules

the alternative that is ranked first by the most voters will be chosen.[2] Analogously, *antiplurality rule* r^A selects the alternative that is ranked last by the fewest voters. *Borda rule* r^B requires each voter i to give $m-1, m-2, \ldots, 0$ points to the alternative that he or she ranks first, second, etc. These points $b_i(a, \mathbf{P}) := \left|\{a' \in A \mid aP_i a'\}\right|$ equal the number of alternatives that i ranks below a. The alternative with the highest total number of points, known as its *Borda score*, is selected. *Copeland rule* r^C considers pairwise majority votes between the alternatives. They define the *majority relation* $a \succ_M^{\mathbf{P}} a' \;:\Leftrightarrow\; \left|\{i \in N \mid aP_i a'\}\right| > \left|\{i \in N \mid a' P_i a\}\right|$ and the alternative that beats the most others according to $\succ_M^{\mathbf{P}}$ is selected.

Copeland rule belongs to the family of *Condorcet methods*: if some alternative a is a Condorcet winner, i.e., beats all others, then $r^C(\mathbf{P}) = a$. This is not true for r^A, r^B and r^P. They are *positional* or *scoring rules*: winners can be characterized as maximizers of scores derived from alternatives' positions in $\mathbf{P}$ and a suitable *scoring vector* $\mathbf{s} \in \mathbb{Z}^m$ with $s_1 \geq s_2 \geq \ldots \geq s_m$. Specifically, let the fact that alternative a is ranked at the j-th highest position in ordering P_i contribute s_j points for a, and refer to the sum of all points received as a's *score*. Then score maximization for $\mathbf{s}^B = (m-1, m-2, \ldots, 1, 0)$ yields the Borda winner, $\mathbf{s}^P = (1, 0, \ldots, 0, 0)$ the plurality winner, and $\mathbf{s}^A = (0, 0, \ldots, 0, -1)$ or $(1, 1, \ldots, 1, 0)$ the antiplurality winner.

We assume that whenever there is a non-singleton set $A^* = \{a^*_{i_1}, \ldots, a^*_{i_k}\}$ of optimizers in Table 1, the alternative $a^*_{i^*} \in A^*$ with lowest index $i^* = \min\{i_1, \ldots, i_k\}$ is selected. This amounts to *lexicographic tie breaking* for $A \subset \{a, \ldots, z, aa, ab, \ldots\}$ and has computational advantages over working with set-valued choices. In particular, only $m^{(m!^n)}$ distinct mappings from preference profiles to alternatives a^* need to be considered,

[2] The formal structure of a committee game is unaffected by whether voting is sincere or strategic. The difference only lies in the interpretation of $\mathcal{P}(A)^n$: it refers to profiles of true preferences in the former and stated ones in the latter case. It is hence without loss of generality if we adopt the simpler vocabulary of sincere voting and say "ranked first" instead of "named as top-ranked".

compared to $(2^m-1)^{(m!^n)}$ if each profile were mapped to a non-empty set $A^* \subseteq A$. The former entails no loss of information as we consider all $\mathbf{P} \in \mathcal{P}(A)^n$: the set of alternatives tied at $\mathbf{P}$ is fully determined by $a^* = r(\mathbf{P})$ and the respective winners $a^{**}, a^{***}, \ldots$ at profiles $\mathbf{P}', \mathbf{P}'', \ldots$ that swap a^* with the alternatives $a', a'', \ldots$ that might be tied with a^* at $\mathbf{P}$.[3] The considered rules r^A, r^B, r^C, r^P and their set-valued versions are hence in one-to-one correspondence and exhibit the same structural equivalences.[4]

3.3 Weighted committee games

Committee games (N, A, ρ) that model real committees, councils, parliaments etc. are more likely than not to involve a non-anonymous social choice rule ρ. First, designated members sometimes possess veto rights or other procedural privileges. Or, second, an anonymous decision rule r may apply not at the level of voters but their respective shareholdings, IMF drawing rights, etc. Third, we can take the relevant players $i \in N$ in a committee game to be well-disciplined parties, factions, or interest groups with different numbers of seats. Anonymity of the underlying rule at the level of individual voters then is destroyed at the level of voter blocs.

In the latter two cases – individual voters with different numbers of votes and groups of voters who act as monolithic blocs – the corresponding rule ρ can be viewed as the *combination of an anonymous social choice rule r with integer voting weights* $w_1, \ldots, w_n$ attached to the relevant players. This yields the most natural kind of weighted committee games. However, also procedural privileges may translate into voting weights implicitly. The textbook case is veto power of the five permanent members in the UN Security Council: the corresponding 15-player (binary) weighted voting game equals $[39; 7, 7, 7, 7, 7, 1, \ldots, 1]$.

In the following, we let r denote the entire family of mappings from n-tuples of linear orders over $A = \{a_1, \ldots, a_m\}$ to winners $a^* \in A$ determined by the considered rule (for all n and m). Then the indicated combination operation amounts to a simple replication. It defines the social choice rule $r|\mathbf{w}\colon \mathcal{P}(A)^{w_\Sigma} \to A$ by

$$r|\mathbf{w}(\mathbf{P}) := r(\underbrace{P_1, \ldots, P_1}_{w_1 \text{ times}}, \underbrace{P_2, \ldots, P_2}_{w_2 \text{ times}}, \ \ldots, \ \underbrace{P_n, \ldots, P_n}_{w_n \text{ times}}) \tag{1}$$

[3] Given $r(\mathbf{P}) = b$, for example, a tie with a can directly be ruled out; one sees if b was tied with c by checking whether $r(\mathbf{P}') = c$ or b where $\mathbf{P}'$ only swaps b's and c's position in every player's ranking P_i.

[4] Analogous reasoning would apply if ties were broken in a uniform random way, i.e., for the most basic type of probabilistic social choice. See Brandl et al. (2016) on differences between deterministic and probabilistic frameworks.

P_1	P_2	P_3	P_4
d	b	c	c
e	c	e	b
b	e	a	a
a	a	d	d
c	d	b	e

$\Rightarrow$

$r^A|\mathbf{w}(\mathbf{P}) = a$ (a has min. bottom ranks 0)
$r^B|\mathbf{w}(\mathbf{P}) = b$ (b has max. Borda score 28)
$r^C|\mathbf{w}(\mathbf{P}) = c$ (c has max. pairwise wins 3)
$r^P|\mathbf{w}(\mathbf{P}) = d$ (d has max. plurality tally 5)

Table 2: Choices for preference profile $\mathbf{P}$ when $\mathbf{w} = (5, 3, 2, 2)$

for a given anonymous rule r and a non-negative, non-degenerate weight vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}_0^n$ with $w_\Sigma := \sum_{i=1}^n w_i > 0$. In the degenerate case $\mathbf{w} = (0, \ldots, 0)$, let $r|\mathbf{0}(\mathbf{P}) \equiv a_1$.

We say a committee game (N, A, ρ) is *r-weighted* for a given anonymous social choice rule r if there exists a weight vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}_0^n$ such that

$$\rho(\mathbf{P}) = r|\mathbf{w}(\mathbf{P}) \text{ for all } \mathbf{P} = (P_1, \ldots, P_n) \in \mathcal{P}(A)^n. \tag{2}$$

Then – so when $(N, A, \rho) = (N, A, r|\mathbf{w})$ – we refer to $(N, A, r, \mathbf{w})$ as a *(weighted) representation* of (N, A, ρ). The corresponding game will also be denoted by $[N, A, r, \mathbf{w}]$.

If the anonymous rule in question is plurality rule r^P, we call $(N, A, r^P|\mathbf{w})$ a *(weighted) plurality committee*. Similarly, $(N, A, r^A|\mathbf{w})$, $(N, A, r^B|\mathbf{w})$, and $(N, A, r^C|\mathbf{w})$ are referred to as an *antiplurality committee*, *Borda committee*, and *Copeland committee*. That such committees can crucially differ for a fixed distribution $\mathbf{w}$ is illustrated in Table 2: the winning alternative all depends on the voting rule r in use.[5]

4 Equivalence classes of weighted committee games

4.1 Equivalence of committee games

Weighted representations of given committee games are far from unique. Consider, e.g., the j-dictatorship game (N, A, ρ_j) where $\rho_j(\mathbf{P})$ equals the alternative that is top-ranked by P_j for every $\mathbf{P} \in \mathcal{P}(A)^n$. This, e.g., coincides with $[N, A, r, \mathbf{w}]$ for $r \in \{r^C, r^P\}$ and any $\mathbf{w} \in \mathbb{N}_0^n$ with $w_j > \sum_{i \neq j} w_i$.

[5] Moreover, e wins under *approval voting* for suitable ballots (Brams and Fishburn 1978). See Felsenthal et al. (1993), Leininger (1993), or Tabarrok and Spector (1999) for related case studies.

Committees $(N, A, r|\mathbf{w})$ and $(N', A', r'|\mathbf{w}')$ evidently are *equivalent* if $N = N'$, $A = A'$, and $r \neq r'$ or $\mathbf{w} \neq \mathbf{w}'$ but the respective mapping from preference profiles to outcomes a^* is the same; that is, when $r|\mathbf{w}(\mathbf{P}) = r'|\mathbf{w}'(\mathbf{P})$ for all $\mathbf{P} \in \mathcal{P}(A)^n$. We here focus on situations where $r = r'$ and try to capture structural equivalence in the sense that $(N, A, r|\mathbf{w})$ and $(N', A', r|\mathbf{w}')$ *reflect the same decision environment* even though weights and labels of players or alternatives might differ. The latter means there are bijections $\pi\colon N \to N'$ and $\tilde{\pi}\colon A \to A'$ such that each player $i \in N$ and alternative $a \in A$ has the same role in $(N, A, r|\mathbf{w})$ as do player $\pi(i)$ and alternative $\tilde{\pi}(a)$ in $(N', A', r|\mathbf{w}')$. Accordingly, r-weighted committee games $(N, A, r|\mathbf{w})$ and $(N', A', r|\mathbf{w}')$ will be called *structurally equivalent* or *equivalent up to isomorphism* if

$$\Big\{a_j \, P_i \, a_k \Leftrightarrow \tilde{\pi}(a_j) \, P'_{\pi(i)} \, \tilde{\pi}(a_k)\Big\} \;\Rightarrow\; \tilde{\pi}\Big(r|\mathbf{w}(\mathbf{P})\Big) = r|\mathbf{w}'(\mathbf{P}') \tag{3}$$

for suitable functions $\pi\colon N \to N'$ and $\tilde{\pi}\colon A \to A'$ that map every profile $\mathbf{P}$ of preferences P_i over A to a relabeled profile $\mathbf{P}'$ of preferences $P'_{\pi(i)}$ over A'.

This includes situations where $N = N'$ but weights $\mathbf{w}'$ are a permutation of $\mathbf{w}$. For instance, the Copeland committee $(N, A, r^C|\mathbf{w})$ has different attractiveness to a given player for $\mathbf{w} = (3, 1, 1)$, $(1, 3, 1)$, or $(1, 1, 3)$. However, the decision environment is the same: it involves a dictator player whose most-preferred alternative wins and two null players whose preferences do not influence the outcome.

A given weight distribution $\mathbf{w} \in \mathbb{N}_0^n$ fixes the number of players. So as labels of players and alternatives do not matter, we write $(r, \mathbf{w}) \sim_m (r, \mathbf{w}')$ to denote that r-committee games with m alternatives are structurally equivalent for weight distributions $\mathbf{w}$ and $\mathbf{w}'$. Relation $\sim_m$ and a suitable reference distribution $\bar{\mathbf{w}} \in \mathbb{N}_0^n$ with $\bar{w}_1 \geq \bar{w}_2 \geq \ldots \geq \bar{w}_n$ serving as index define the *equivalence class*

$$\mathcal{E}^r_{\bar{\mathbf{w}},m} := \Big\{\mathbf{w} \in \mathbb{N}_0^n \mid (r, \mathbf{w}) \sim_m (r, \bar{\mathbf{w}})\Big\}. \tag{4}$$

$\mathcal{E}^r_{\bar{\mathbf{w}},m}$ is the set of all weight distributions that give rise to weighted committee games equivalent to $[N, A, r, \bar{\mathbf{w}}]$ up to isomorphism. If voters use rule r for deciding between m alternatives, then all weight distributions $\mathbf{w}, \mathbf{w}' \in \mathcal{E}^r_{\bar{\mathbf{w}},m}$ come with identical monotonicity properties, voting paradoxes, manipulation incentives, strategic equilibria, implementation possibilities, etc.

4.2 Illustration

As an example, consider Borda rule r^B for $m = 3$ and reference weights $\bar{\mathbf{w}} = (5, 2, 1)$. We focus on the subset $\mathcal{E}^B_{(5,2,1),3} \subset \mathcal{E}^{r^B}_{(5,2,1),3}$ of vectors $\mathbf{w}$ with $w_1 \geq w_2 \geq w_3$. Identity of $\rho = r^B|(5, 2, 1)$ and $r^B|\mathbf{w}$ implies two inequalities for each profile $\mathbf{P} \in \mathcal{P}(A)^3$: the Borda winner must beat each of the other alternatives. Writing *abc* in abbreviation of aP_ibP_ic, profile $\mathbf{P} = (cab, bac, abc)$, for instance, implies $r^B|\bar{\mathbf{w}}(\mathbf{P}) = c$ and hence the Borda score of (lexicograpically maximal) c under any equivalent weight vector $\mathbf{w}$ must strictly exceed that of a and b:

$$2w_1 > w_1 + w_2 + 2w_3 \tag{I}$$

$$2w_1 > 2w_2 + w_3. \tag{II}$$

$\mathbf{P}' = (cab, abc, bac)$ makes a the winner. Its score must not be smaller than b's and c's:

$$w_1 + 2w_2 + w_3 \geq w_2 + 2w_3 \tag{III}$$

$$w_1 + 2w_2 + w_3 \geq 2w_1. \tag{IV}$$

Profiles $\mathbf{P}'' = (abc, bca, bac)$ and $\mathbf{P}''' = (abc, bca, bca)$ similarly induce $\rho(\mathbf{P}'') = a$ and $\rho(\mathbf{P}''') = b$, which implies

$$2w_1 + w_3 \geq w_1 + 2w_2 + 2w_3 \tag{V}$$

$$2w_1 + w_3 \geq w_2 \tag{VI}$$

$$w_1 + 2w_2 + 2w_3 > 2w_1 \tag{VII}$$

$$w_1 + 2w_2 + 2w_3 \geq w_2 + w_3. \tag{VIII}$$

Condition (VIII) is trivially satisfied for any $\mathbf{w} \in \mathbb{N}_0^n$. (IV) and (V) imply $w_1 = 2w_2+w_3$. This makes (I) equivalent to $w_2 > w_3$ and (VII) to $w_3 > 0$. Combining $w_1 = 2w_2 + w_3$ and $w_2 > w_3 > 0$ also verifies (II), (III) and (VI). The 212 remaining profiles $\mathbf{P} \in \mathcal{P}(A)^3$ turn out not to impose additional constraints. Hence

$$\mathbf{w} \in \mathcal{E}^B_{(5,2,1),3} = \Big\{(2w_2 + w_3, w_2, w_3) \in \mathbb{N}_0^3 \,:\, w_2 > w_3 > 0\Big\}. \tag{5}$$

The full class $\mathcal{E}^{r^B}_{(5,2,1),3}$ follows by permuting the weight distributions in $\mathcal{E}^B_{(5,2,1),3}$. Other equivalence classes, such as $\mathcal{E}^{r^B}_{(1,1,1),3}$, $\mathcal{E}^{r^B}_{(2,1,1),3}$, etc., can be characterized analogously. However, determining *all classes* is quite involved even for $n = m = 3$: there exist $3^{(3!^3)} = 3^{216} > 10^{103}$ different mappings from preferences to outcomes that, in principle, have to be checked for possessing a weighted Borda representation (compared to just $\approx 10^{80}$ atoms in the universe).

4.3 Relation between equivalence classes

As the number of distinct mappings from preference profiles to outcomes is finite for given n and m, there are only finitely many disjoint $\mathcal{E}^r_{\bar{\mathbf{w}},m}$ with $\bar{\mathbf{w}} \in \mathbb{N}_0^n$ for any given rule r. They partition the infinite space $\mathbb{N}_0^n$ of weight distributions into a collection $\left\{\mathcal{E}^r_{\bar{\mathbf{w}}_1,m}, \mathcal{E}^r_{\bar{\mathbf{w}}_2,m}, \ldots, \mathcal{E}^r_{\bar{\mathbf{w}}_\xi,m}\right\}$ of all r-weighted committees with n voters deciding on m alternatives. We will see below that the numbers ξ of elements of such a partition – hence the numbers of structurally distinct weighted committee games for given r, n, and m – vary widely across rules.

Let us first derive some analytical (non-computational) results on the relation between equivalence classes for different rules r or parameters n and m; proofs are provided in Appendix A. The degenerate weight vector $\mathbf{w}^0 = \mathbf{0}$ always forms its own equivalence class:

Lemma 1. *Let $m \geq 2$, $r \in \{r^A, r^B, r^C, r^P\}$ and $\mathbf{w} \neq \mathbf{0} \in \mathbb{N}_0^n$. Then $(r, \mathbf{0}) \nsim_m (r, \mathbf{w})$.*

We focus on non-degenerate weight vectors $\mathbf{w} \neq \mathbf{0}$ from now on. Another straightforward observation is that the considered rules do not differ for $m = 2$:

Proposition 1. *The partitions $\left\{\mathcal{E}^r_{\bar{\mathbf{w}}_1,2}, \ldots, \mathcal{E}^r_{\bar{\mathbf{w}}_\xi,2}\right\}$ of $\mathbb{N}_0^n$ coincide for $r \in \{r^A, r^B, r^C, r^P\}$.*

Furthermore, weighted committee games with $m = 2$ are in one-to-one relation to standard weighted voting games $[q; w_1, \ldots, w_n]$ with a 50%-majority quota:

Proposition 2. *Let $N = \{1, \ldots, n\}$ and $A = \{a_1, a_2\}$. For any $\mathbf{w} \neq \mathbf{0} \in \mathbb{N}_0^n$ and $r \in \{r^A, r^B, r^C, r^P\}$*

$$r|\mathbf{w}(\mathbf{P}) = a_1 \Leftrightarrow v(S) = 1$$

where v is the coalitional function of weighted voting game $(N, v) = [q; \mathbf{w}]$ with $q = \frac{1}{2}\sum_{i\in N} w_i$ and coalition $S = \{i \in N \mid a_1\, P_i\, a_2\} \subseteq N$ collects all players who prefer a_1 at profile $\mathbf{P} \in \mathcal{P}(A)^n$.

It follows that the respective partitions $\left\{\mathcal{E}^r_{\bar{\mathbf{w}}_1,2}, \ldots, \mathcal{E}^r_{\bar{\mathbf{w}}_\xi,2}\right\}$ of $\mathbb{N}_0^n$ coincide with those for weighted voting games with a simple majority quota. Their study and enumeration for $n \leq 5$ dates back to von Neumann and Morgenstern (1953, Ch. 10).

The remaining propositions consider equivalence classes for a fixed rule r as the number m of alternatives is varied.

Proposition 3. *For Copeland rule r^C, the partitions $\left\{\mathcal{E}^{r^C}_{\bar{\mathbf{w}}_1,m}, \dots, \mathcal{E}^{r^C}_{\bar{\mathbf{w}}_\xi,m}\right\}$ of $\mathbb{N}_0^n$ coincide for all $m \geq 2$.*

Proposition 4. *For plurality rule r^P, the partitions $\left\{\mathcal{E}^{r^P}_{\bar{\mathbf{w}}_1,m}, \dots, \mathcal{E}^{r^P}_{\bar{\mathbf{w}}_\xi,m}\right\}$ of $\mathbb{N}_0^n$ coincide for all $m \geq n$.*

Proposition 5. *For Borda rule r^B and given $m \geq 3$, every weight vector $\tilde{\mathbf{w}}_\mathbf{j} = (j, 1, 0, \dots, 0)$ with $j \in \{1, \dots, m-1\}$ identifies a different class $\mathcal{E}^{r^B}_{\tilde{\mathbf{w}}_\mathbf{j},m}$.*

It follows that for any fixed number of players, the number ξ of structurally distinct Borda committee games grows without bound as m goes to infinity. Borda rule differs in this respect from Copeland, plurality, and also antiplurality plurality rule:

Proposition 6. *For antiplurality rule r^A, the partitions $\left\{\mathcal{E}^{r^A}_{\bar{\mathbf{w}}_1,m}, \mathcal{E}^{r^A}_{\bar{\mathbf{w}}_2,m}, \dots, \mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\xi,m}\right\}$ of $\mathbb{N}_0^n \setminus \{\mathbf{0}\}$ consist of $\xi = n$ equivalence classes identified by weight vectors $\bar{\mathbf{w}}_1 = (1, 0, \dots, 0), \bar{\mathbf{w}}_2 = (1, 1, \dots, 0), \dots, \bar{\mathbf{w}}_\mathbf{n} = (1, 1, \dots, 1)$ for all $m \geq n + 1$.*

The reference vectors in Propositions 5 and 6 have the lowest possible weight sum for the respective class of games. Before we elaborate on this be reminded that equivalence classes would be the same if we considered uniform tie breaking or set-valued choices (cf. end of Section 3.2).[6]

5 Computational identification of weighted committees

5.1 Minimal representations and test for weightedness

Above rules have the property that $[N, A, r, \mathbf{w}] = [N, A, r, \mathbf{w}']$ when $\mathbf{w}$ is a multiple of $\mathbf{w}'$. Even if $\mathbf{w}$ represents the actual distribution of seats or vote shares in a given institution, it can be analytically more convenient to work with $\mathbf{w}'$. More generally, given $(N, A, \rho) = (N, A, r|\mathbf{w})$, we say that $(N, A, r, \mathbf{w})$ has *minimum integer sum* or is a *minimal representation* of (N, A, ρ) if $\sum_{i \in N} w'_i \geq \sum_{i \in N} w_i$ for all representations $(N, A, r, \mathbf{w}')$ of (N, A, ρ) that involve rule r. The games in a given equivalence class $\mathcal{E}^r_{\bar{\mathbf{w}},m}$ usually have a unique minimal representation.[7] The corresponding minimal

[6]Lexicographic tie breaking can yield $r|\mathbf{w}(\mathbf{P}) = r|\mathbf{w}'(\mathbf{P}) = a^*$ even though the sets of alternatives tied at $\mathbf{P}$, say A^* and A'^*, differ between $r|\mathbf{w}$ and $r|\mathbf{w}'$. Then construct $\mathbf{P}'$ as follows: if $A^* \not\subset A'^*$, fix an alternative $a' \in A^* \setminus A'^*$ and swap positions of a^* and a' in $\mathbf{P}$. Now $r|\mathbf{w}(\mathbf{P}') = a^*$ is unchanged but $r|\mathbf{w}'(\mathbf{P}') \neq a^*$. If $A^* \subset A'^*$, consider $a' \in A'^* \setminus A^*$ analogously.

[7]If $m = 2$, minimal representations are unique for up to $n = 7$ players (Kurz 2012). Multiplicities for games with larger values of m or n arise but are rare.

weights are the focal choice for $\bar{\mathbf{w}}$. For instance, $(5,2,1)$ has minimal sum among all $\mathbf{w} \in \mathcal{E}^{r^B}_{(5,2,1),3}$ characterized in Section 4.2.

Finding minimal representations of arbitrary Copeland committees simplifies to finding them for $m = 2$ by Proposition 3. And by Proposition 2 this amounts to finding minimal representations of specific weighted voting games. Linear programming techniques have proven helpful for this task and can be adapted to committees that involve scoring rules r^A, r^B, or r^P. Namely, for an arbitrary scoring rule r that induces social choice rule ρ for appropriate weights, let us denote the index of the winning alternative at profile $\mathbf{P}$ by $\omega_\rho(\mathbf{P}) \in \{1, \ldots, m\}$, i.e., $\rho(\mathbf{P}) = a_{\omega_\rho(\mathbf{P})} \in A$; write $S_k(P_i) \in \mathbb{Z}$ for the unweighted $(s_1, s_2, \ldots, s_m)$-score of alternative a_k derived from its position in ordering P_i (e.g., for $m = 3$ and $a_3 = c$, we have $S_3(P_i) = s_2$ if either aP_icP_ib or bP_icP_ia). Then any solution to the following *integer linear program* yields a minimal representation $(N, A, r, \mathbf{w})$ of (N, A, ρ):

$$\min_{\mathbf{w} \in \mathbb{N}_0^n} \sum_{i=1}^{n} w_i \tag{ILP}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} S_k(P_i) \cdot w_i \leq \sum_{i=1}^{n} S_{\omega_\rho(\mathbf{P})}(P_i) \cdot w_i - 1 \quad \forall \mathbf{P} \in \mathcal{P}(A)^n \, \forall 1 \leq k \leq \omega_\rho(\mathbf{P}) - 1,$$

$$\sum_{i=1}^{n} S_k(P_i) \cdot w_i \leq \sum_{i=1}^{n} S_{\omega_\rho(\mathbf{P})}(P_i) \cdot w_i \quad \forall \mathbf{P} \in \mathcal{P}(A)^n \, \forall \omega_\rho(\mathbf{P}) + 1 \leq k \leq m.$$

The case distinction between scores of non-winning alternatives a_k with index $k < \omega_\rho(\mathbf{P})$ vs. $k > \omega_\rho(\mathbf{P})$ reflects the tie breaking assumption. If some (non-minimal) representation $(N, A, r, \mathbf{w}')$ of (N, A, ρ) is known and $w'_1 \geq w'_2 \geq \ldots \geq w'_n$ then adding constraints $w_i \geq w_{i+1}$, $\forall 1 \leq i \leq n - 1$, to (ILP) helps to speed up computations.

If it is not yet known whether ρ is r-weighted, (ILP) provides a decisive *test for r-weightedness* for any scoring rule r.[8] Namely, the constraints in (ILP) characterize a non-empty compact set if and only if ρ is r-weighted. Checking non-emptiness of the constraint set for a given ρ answers the question of its r-weightedness. This can be done with optimization software (e.g., Gurobi or CPLEX) that also determines a weight sum minimizer at little extra effort.

[8]This extends to Copeland rule r^C by Propositions 1 and 3.

Branch-and-Cut Algorithm	
Given n, m and r, identify every class $\mathcal{E}^r_{\bar{\mathbf{w}}_\mathbf{k},m}$ by a minimal representation.	
Step 1	Generate all $J := (m!)^n$ profiles $\mathbf{P}^1, \ldots, \mathbf{P}^J \in \mathcal{P}(A)^n$ for $A := \{a_1, \ldots, a_m\}$. Set $\mathcal{F} := \emptyset$.
Step 2	For every $\mathbf{P}^j \in \mathcal{P}(A)^n$ and every $a_i \in A$, check if there is any weight vector $\mathbf{w} \in \mathbb{N}_0^n$ s.t. $r|\mathbf{w}(\mathbf{P}^j) = a_i$ by testing feasibility of the implied constraints (cf. Section 4.2). If yes, then append (i, j) to $\mathcal{F}$.
Step 3	Loop over j from 1 to J.
Step 3a	If $j = 1$, then set $C_1 := \{1 \le i \le m \mid (i, j) \in \mathcal{F}\}$.
Step 3b	If $j \ge 2$, then set $C_j := \emptyset$ and loop over all $(p_1, \ldots, p_{j-1}) \in C_{j-1}$ and all $p_j \in \{1, \ldots, m\}$ with $(p_j, j) \in \mathcal{F}$. If (ILP) has a solution for the restriction to the profiles $\mathbf{P^1}, \ldots, \mathbf{P^j}$ with prescribed winners $\rho(\mathbf{P^i}) = a_{p_i}$ for $1 \le i \le j$, then append $(p_1, \ldots, p_j)$ to C_p.
Step 4	Loop over the elements $(p_1, \ldots, p_j, \ldots, p_J) \in C_J$ and output minimal weights $\bar{\mathbf{w}}$ such that $r|\bar{\mathbf{w}} \equiv \rho$ with $\rho(\mathbf{P^j}) = p_j$ by solving (ILP).

Table 3: Determining the classes of r-weighted committees for given n and m

5.2 Algorithm for identifying all r-committees

We would like to characterize all r-committee games for fixed n and m. In principle, one could do this as follows: loop over the $m^{(m!^n)}$ different social choice rules $\rho\colon \mathcal{P}(A)^n \to A$; conduct above test for r-weightedness; if it was successful, determine a representation $(N, A, r, \bar{\mathbf{w}})$ and characterize $\mathcal{E}^r_{\bar{\mathbf{w}},m}$ as in Section 4.2; continue until all rules ρ have been covered.

The extreme growth of $m^{(m!^n)}$ prevents a direct implementation of this idea. However, many mappings can be dropped from consideration in large batches. If $\rho(\mathbf{P}) = a_1$ for one of the $(m-1)!^n$ profiles $\mathbf{P}$ where a_1 is unanimously ranked last, for instance, then ρ cannot be r-weighted for $r \in \{r^A, r^B, r^C, r^P\}$. This rules out $m^{(m!^n-1)}$ candidate mappings in one go. Similarly, if weights $\mathbf{w}$ such that $r|\mathbf{w}(\mathbf{P}) = a_1$ turn out to be incompatible with $r|\mathbf{w}(\mathbf{P}') = a_2$ for two suitable profiles $\mathbf{P}, \mathbf{P}'$, then all $m^{(m!^n-2)}$ mappings ρ with $\rho(\mathbf{P}) = a_1$ and $\rho(\mathbf{P}') = a_2$ can be disregarded at once.

The branch-and-cut algorithm described in Table 3 operationalizes these considerations. Alas, it can still require impractical memory size and running time. The main alternative then is to loop over different weight distributions and check if they are structurally distinct from those already known. Namely, start with $w_\Sigma := 0$ and an empty list $\hat{\mathcal{W}}$ of weight vectors; increase the sum of weights w_Σ in steps of 1;

r / n,m	Antiplurality	Borda	Copeland	Plurality
3,2	4			
4,2	9			
5,2	27			
6,2	138			
7,2	1 663			
8,2	63 764			
9,2	9 425 479			
3,3	5	51	4	6
3,4	3	505	4	6
3,5	3	$\geq$ 2 251	4	6
4,3	19	5 255	9	34
4,4	7	$\gg$ 635 622	9	36
4,5	4	$\gg$ 635 622	9	36
5,3	263	$\gg$ 1 153 448	27	852
6,3	$\geq$ 33 583	$\gg$ 1 153 448	138	$\gg$ 147 984

Table 4: Numbers of distinct weighted committee games

generate the set $\mathcal{W}_{w_\Sigma} := \left\{\mathbf{w} \in \mathbb{N}_0^n \,\middle|\, w_1 \geq \cdots \geq w_n \text{ and } w_1 + \cdots + w_n = w_\Sigma\right\}$ and loop over all $\mathbf{w} \in \mathcal{W}_{w_\Sigma}$. The respective weight vector $\mathbf{w}$ is appended to $\hat{\mathcal{W}}$ if for every $\mathbf{w}' \in \hat{\mathcal{W}}$ we have $r|\mathbf{w}(\mathbf{P}) \neq r|\mathbf{w}'(\mathbf{P})$ for at least one $\mathbf{P} \in \mathcal{P}(A)^n$. The set $\hat{\mathcal{W}}$ then contains a growing list of minimal weight vectors that correspond to structurally distinct committee games $[N, A, r, \mathbf{w}]$. This method has the advantage of not requiring any weightedness test, such as (ILP). However, search needs to be stopped manually and just produces a lower bound on the actual number of classes.[9]

6 Number and geometry of weighted committee games

6.1 Number of antiplurality, Borda, Copeland, and plurality games

A combination of our analytical findings and computational methods permits us to identify all structurally distinct r-weighted committee games with $r \in \{r^A, r^B, r^C, r^P\}$ for small n and m. This can be useful in several ways: demonstrating, for instance, that a certain voting paradox does not occur for any of the 34 distinct plurality committees with $n = 4$, $m = 3$, which we list in Appendix B, would suffice to establish that at least five voter groups or four alternatives are needed for r^P to exhibit the paradox. A characterization of voting equilibria for, say, the 7 weight vectors listed for antiplurality rule when $n = m = 4$ would automatically extend to all distributions of votes, etc.

Table 4 summarizes our findings. Figures do not include the degenerate class $\mathcal{E}_{\mathbf{0},m}$. Numbers for $m = 2$ and $n \leq 6$ have been determined in the literature before; all others are new results (to the best of our knowledge). When less than 150 equivalence classes of games exist, we report a minimum sum integer representation for each in Appendix B.[10] The list for $m = 2$ nests the weighted voting games with 50%-majority threshold reported by Krohn and Sudhölter (1995) and Brams and Fishburn (1996); plurality committees with $m = 3$ nest the subset of tie-free games listed by Chua et al. (2002) for $n = 3, 4$. The branch-and-cut approach required excessive memory for Borda committees when $m > 4$ or $n = m \geq 4$. We write "$\geq \ldots$" if the search algorithm appended no new games to set $\hat{\mathcal{W}}$ for long enough to support the conjecture that the respective bound equals the exact number of games; we write "$\gg \ldots$" if we expect more computing power to yield higher numbers.[11]

6.2 Geometry of committee games with n=3

In principle, one could characterize the full equivalence class of committee games for each reference distribution that we list in the appendix. We have indicated how in Section 4.2. But computation of the respective partition of $\mathbb{N}_0^n$ is very arduous – much more than determining into which classes given games $[N, A, r, \mathbf{w}]$ fall.

[9] One can compute upper bounds on the weight sum that guarantees coverage of *all* equivalence classes, analogously to bounds for minimal representation of weighted voting games (see Muroga 1971, Thm. 9.3.2.1). In our context such bounds are way too large to be practical, however.

[10] When there are less than a million classes, representations will be made available on our websites.

[11] We used 128 GB RAM and eight 3.0 GHz cores. Some instances ran for more than six months.

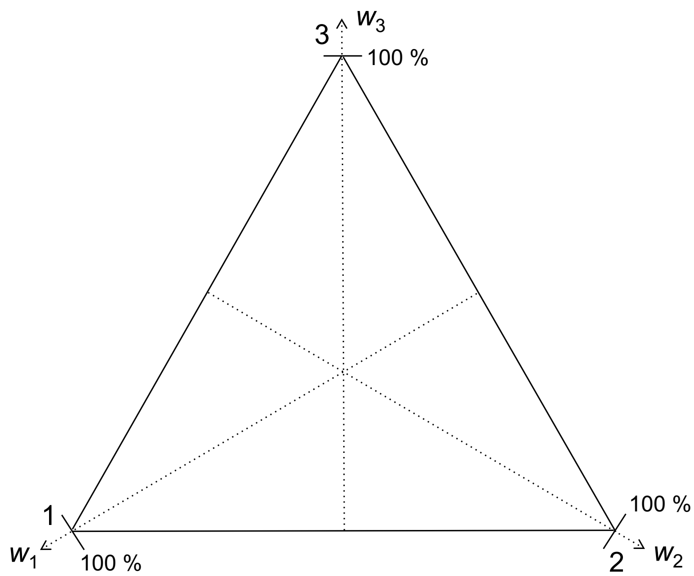

Figure 1: Simplex of all distributions of relative voting weights for $n = 3$

We have done the latter to obtain a first overview of the geometry of committee games. Our illustrations differ in content but echo the geometric approach to voting espoused by Saari (1995, 2001). His eponymous triangles concern $m = 3$ alternatives and consider an arbitrary number n of voters. They illuminate how collective rankings vary with the applicable voting procedure for given preferences.

We, by contrast, assume $n = 3$ voter blocs and let the number m of alternatives vary. Points in our triangles correspond to voting weight distributions; colors group them into equivalence classes. We use the standard projection of the 3-dimensional unit simplex of relative weights to the plane, which is illustrated in Figure 1. The weight axes are suppressed in subsequent figures. Points of identical color correspond to structurally equivalent weight distributions, i.e., they induce isomorphic committee games for the voting rule r under investigation. When equivalence classes correspond to line segments or single points, we have manually enlarged these in Figures 2 and 3 to improve visibility.

6.2.1 Copeland committees

Figure 2(a) shows all Copeland committees with three players. The four equivalence classes $\mathcal{E}^{r^C}_{\bar{\mathbf{w}},m}$ with $\bar{\mathbf{w}} \in \big\{(1,0,0),(1,1,0),(1,1,1),(2,1,1)\big\}$, $m \geq 2$, can be identified as follows. The dark blue triangles in the corners collect all weight distributions in $\mathcal{E}^{r^C}_{(1,0,0),m}$: one group with more than 50% of the votes can impose its preferred

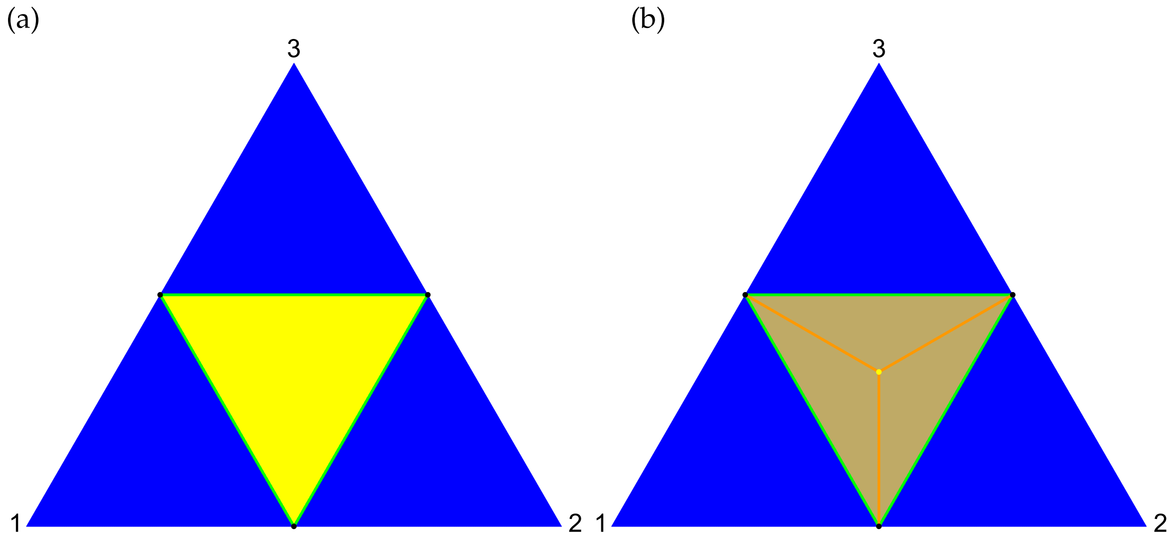

Figure 2: The (a) four Copeland and (b) six plurality equivalence classes

alternative as a dictator. The green lines cover all weight distributions in $\mathcal{E}^{r^C}_{(2,1,1),m}$: one player holds 50% of the votes, the others share the rest in an arbitrary positive proportion. The three black points depict situations in which two players have equal positive numbers of votes while the third has no votes, i.e., $\mathcal{E}^{r^C}_{(1,1,0),m}$. The yellow triangle in the middle reflects the many equivalent weight configurations in $\mathcal{E}^{r^C}_{(1,1,1),m}$: each player wields a positive number of votes below half the total. It is known from the analysis of binary weighted voting games that quite dissimilar distributions like $(33, 33, 33)$ and $(49, 49, 1)$ induce the same pairwise majorities.

6.2.2 Plurality committees

Figure 2(b) illustrates the situation for $m \geq 3$ when plurality rule r^P is used. Weight vectors $\mathbf{w}$ that belong to Copeland class $\mathcal{E}^{r^C}_{(1,1,1),m}$ split into the plurality classes $\mathcal{E}^{r^P}_{(1,1,1),m}$ with identical weights for all three players, $\mathcal{E}^{r^P}_{(2,2,1),m}$ and $\mathcal{E}^{r^P}_{(3,2,2),m}$. The former corresponds to weights on the orange lines that point to the center: two players each have a plurality of votes. The latter class involves just one plurality player.

For non-dictatorial weight configurations, plurality rule is more sensitive to the configuration of seats or voting rights than Copeland rule. This becomes more pronounced the more players are involved: Table 4 shows that there are about four and 32 times more structurally different committees with plurality than Copeland rule for $n = 4$ and 5, respectively; we conjecture this factor exceeds 1 000 for $n = 6$.

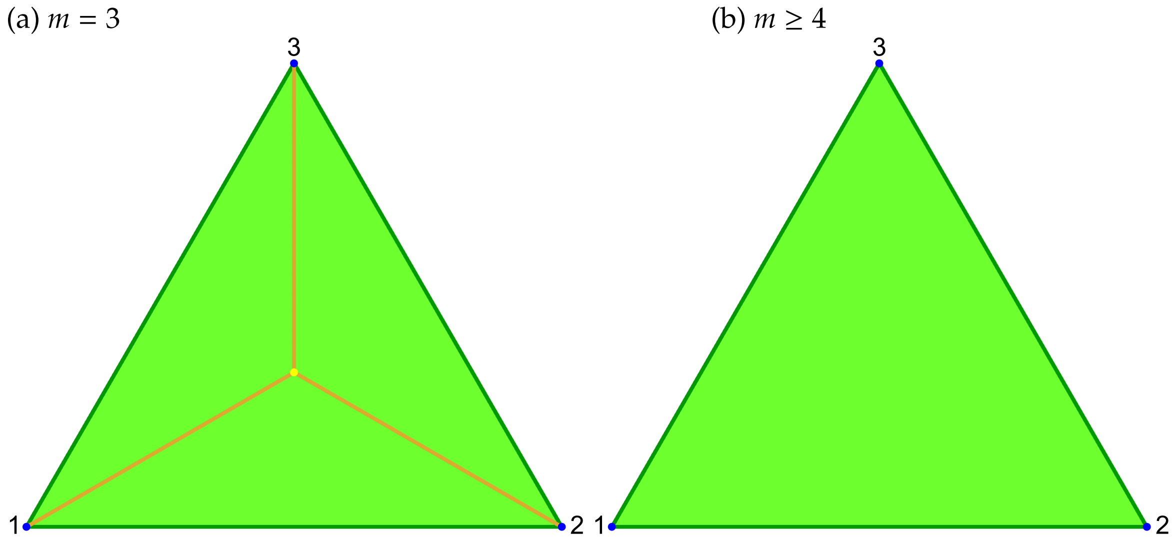

Figure 3: The five or three antiplurality equivalence classes

6.2.3 Antiplurality committees

In Figure 3, the dark blue triangles that reflected existence of a dictator player under r^C and r^P in Figure 2 shrink to the three vertices for antiplurality rule. Only the degenerate case, in which no one else has positive weight, has outcomes determined by one player's preferences alone. Otherwise, even a single vote may disqualify an alternative under r^A.

Equivalence classes $\mathcal{E}^{r^A}_{\bar{\mathbf{w}},3}$ with $\bar{\mathbf{w}} \in \left\{(1,0,0),(1,1,0),(1,1,1),(2,1,1),(2,2,1)\right\}$ differ according to whether one (blue vertices), two (dark green edges), or all three players have positive weight. The latter case comes with the possibility that none (yellow center), one (orange lines), or two of them (light green triangles) have greater weight than others and hence elevated roles if the players each vote against a different alternative. For $m = 4$, this distinction becomes obsolete because there is always at least one alternative not disapproved by anyone (Proposition 6). Then there are just three classes $\mathcal{E}^{r^A}_{\bar{\mathbf{w}},4}$ with $\bar{\mathbf{w}} \in \left\{(1,0,0),(1,1,0),(1,1,1)\right\}$.

6.2.4 Borda committees

Figures 2(a) and 4 show how sensitive Borda decision structures are to the underlying vote distribution – the more alternatives, the higher the sensitivity. (Recall that Figure 2(a) captures the case of $m = 2$ for all rules.) This does not make a big practical difference if preference configurations $\mathbf{P}$ with $r^B|\mathbf{w}(\mathbf{P}) \neq r^B|\mathbf{w}'(\mathbf{P})$ for similar $\mathbf{w}, \mathbf{w}'$ are rare empirically compared to those where $\mathbf{w}$ and $\mathbf{w}'$ result in identical

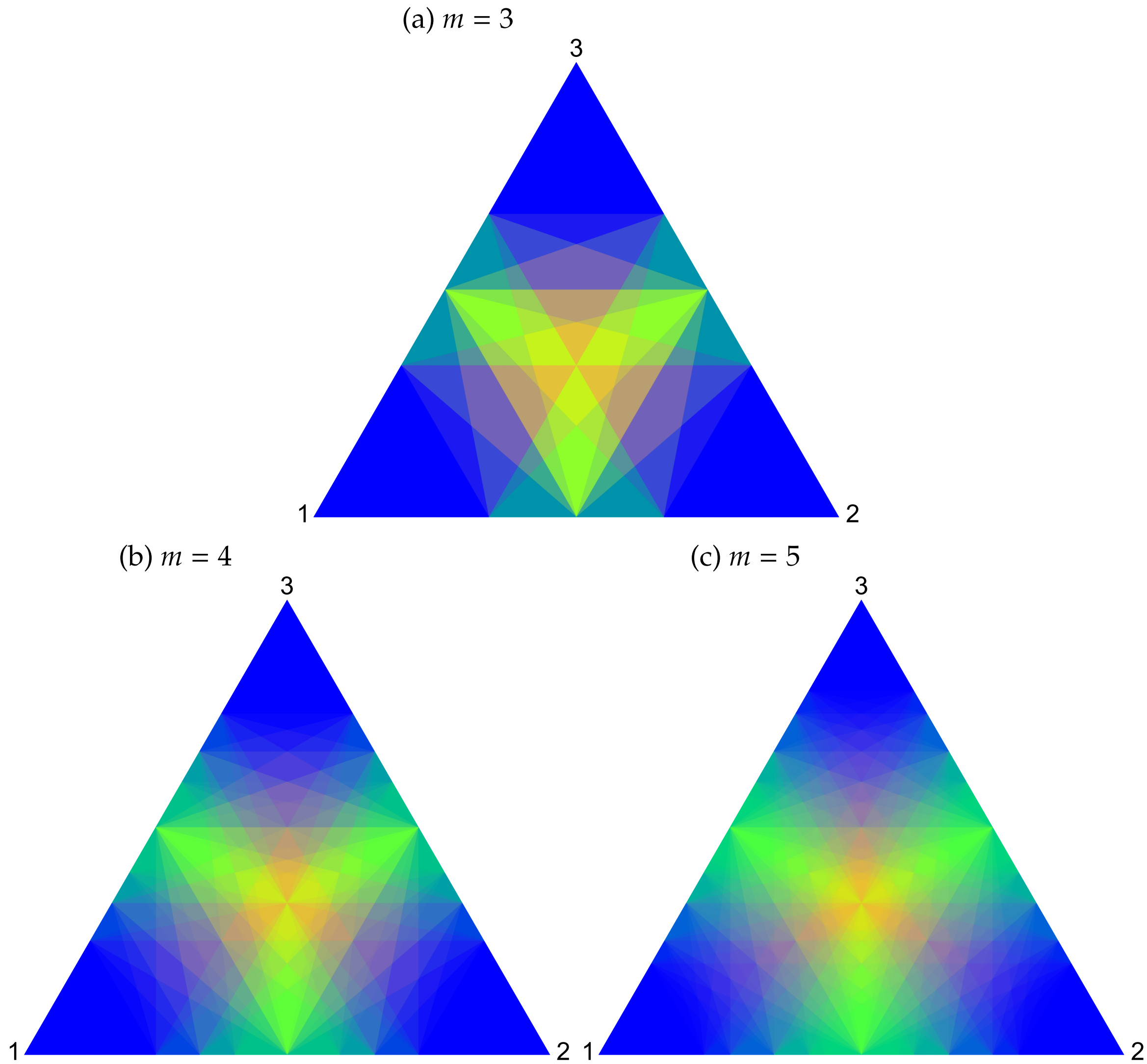

Figure 4: The 51, 505, and ≥2251 Borda equivalence classes

outcomes.[12] But from an a priori perspective, r^B involves more scope for changes in the distribution of voting rights to induce different decisions than r^A, r^C and r^P.

The dark blue triangles in the corners of Figure 4 are smaller than those in Figure 2: having 50% plus one vote suffices to win all pairwise comparisons or plurality votes but more than two thirds are needed under Borda rule.[13]

7 Concluding remarks

Equivalence of different distributions of seats, drawing rights, voting stock, etc. depends highly on whether decisions involve two, three, or more alternatives. Weight distributions such as $(6,5,2)$, $(5,5,5)$, or $(48\%, 24\%, 28\%)$ are equivalent for binary majority choices but not more generally. Scope for weight differences to matter has been formalized and compared across rules in this chapter.

For the IMF's Executive board, we have checked that the 2016 reform of drawing rights can indeed have consequences for who becomes the next IMF Managing Director. Election winners differ between reformed and unreformed weights for about 5% of all 6^{24} conceivable strict preference configurations for three candidates. It matters for up to 20% of profiles whether plurality, plurality with runoff, or Copeland rule is applied.

Copeland rule, as the only Condorcet method featured here, behaves somewhat at odds with the three scoring rules: it extends the equivalences known for dichotomous choice problems to arbitrarily many options (Proposition 3). This might not feel surprising because winners in Copeland games are selected by binary comparisons. Is it, therefore, okay to apply insights and tools for binary voting, such as the Shapley-Shubik or Penrose-Banzhaf power indices, also to voting bodies that face non-binary options as long as the pertinent rules satisfy the Condorcet winner criterion?

This conjecture is wrong. The Copeland method is special insofar that it invokes ordinal evaluations only; most other Condorcet methods also use information on victory margins, rank positions, or distances. More alternatives then generate more scope for decisions to be sensitive to the seat distribution. Proposition 3 fails to generalize, for example, to Black committee games. The *Black rule* selects the Condorcet

[12]Our color choices provide a rough guide to how much two mappings $r|\mathbf{w}$ and $r|\mathbf{w}'$ differ: points of similar color correspond to committees whose decisions differ for few profiles.

[13]Player 1's relative weight must exceed $(m-1)/m$ to be a Borda dictator. This was already observed by Borda in 1781. Moulin (1982) studies a more nuanced concept of veto power for Borda and Copeland rule, which translates to lighter colors in our figures.

winner if one exists and otherwise uses Borda scores to break cyclical majorities. Weight distributions of $(6,4,3)$ and $(4,4,2)$ are equivalent for $m = 2$ and give rise to a cycle $a \succ_M^{\mathbf{P}} b \succ_M^{\mathbf{P}} c \succ_M^{\mathbf{P}} a$ for profile $\mathbf{P} = (cab, abc, bca)$. The Black winner is c for the former weight distribution, with a score of 15; but a wins with a score of 12 for the latter. Hence they are non-equivalent for $m = 3$. The same applies to *Kemeny–Young* or *maximum likelihood rule*, which minimizes total pairwise disagreements (Kemeny distances) between the rankings in profile $\mathbf{P}$ and the collective ranking; or *maximin rule*, where a winner must maximize the minimum support across all pairwise comparisons. There are more distinct Black, Kemeny–Young, or maximin committee games than Copeland ones although all involve Condorcet methods.

There is ample choice for extending the analysis. The list of sensible single-winner voting procedures that could be used by a committee is long (see, e.g., Aleskerov and Kurbanov 1999; Nurmi 2006, Ch. 7; or Laslier 2012). We have tentatively tried to identify the number of distinct committee games that involve scoring rules based on arbitrary $\mathbf{s} = (1, s_2, 0) \in \mathbb{Q}^3$ for $n = m = 3$. The numbers of structurally distinct games are roughly M-shaped: they increase from 6 plurality committees to more than 160 for $s_2 = 0.25$, fall to 51 Borda committees for $s_2 = 0.5$, increase again to at least 229 for $s_2 = 0.9$ and then drop sharply to just 5 antiplurality committees for $s_2 = 1$.

The equivalence of seemingly different committee games is of theoretical and applied interest. It is relevant for the design of actual voting bodies such as the IMF's Executive Board, councils of non-governmental organizations, boards of private companies, and possibly even for empirical analysis and forecasting: sampling errors in opinion poll data should matter less, for instance, when population shares of the relevant groups fall into the middle of a big equivalence class of the applicable election rule than for a boundary point.

Whether sensitivity of collective decisions to weight differences is (un)desirable from an institutional perspective depends on context and objectives. Higher sensitivity can give bigger incentives for parties to campaign or for investment into voting stock. However, this needs to be weighed against other properties of the applicable voting methods such as monotonicity properties, ease of manipulation, etc. Links between the weight distribution and decisions are just one aspect of voting among many – but one that matters beyond binary options.

Appendix A: Proofs

Proof of Lemma 1

Consider $\mathbf{w} \neq \mathbf{0} \in \mathbb{N}_0^n$ and the unanimous profile $\mathbf{P} = (P, \dots, P) \in \mathcal{P}(A)^n$ with $a_2 P a_3 P \dots P a_m P a_1$. Then $r|\mathbf{0}(\mathbf{P}) = a_1$ but $r|\mathbf{w}(\mathbf{P}) = a_2$ for any $r \in \{r^A, r^B, r^C, r^P\}$.

Proof of Proposition 1

For $A = \{a_1, a_2\}$ and arbitrary fixed $\mathbf{w} \neq \mathbf{0} \in \mathbb{N}_0^n$

$$r|\mathbf{w}(\mathbf{P}) = \begin{cases} a_2 & \text{if } \sum\limits_{i:\, a_2 P_i a_1} w_i > \sum\limits_{j:\, a_1 P_j a_2} w_j, \\ a_1 & \text{otherwise} \end{cases}$$

for any $r \in \{r^A, r^B, r^C, r^P\}$. So antiplurality, Borda, Copeland and plurality rule are equivalent and hence have the same equivalence classes.

Proof of Proposition 2

Define $w(T) := \sum_{i \in T} w_i$ for $T \subseteq N$. If $w(S) \geq w(N \smallsetminus S)$ then $r^P|\mathbf{w}(\mathbf{P}) = a_1$ and $v(S) = 1$. If $w(S) < w(N \smallsetminus S)$ then $r^P|\mathbf{w}(\mathbf{P}) = a_2$ and $v(S) = 0$. Proposition 1 extends this to $r \in \{r^A, r^B, r^C\}$.

Proof of Proposition 3

For a given set of alternatives $A = \{a_1, \dots, a_m\}$ and any subset $A' \subseteq A$ that preserves the order of the alternatives, we denote the *projection* of preference profile $\mathbf{P} \in \mathcal{P}(A)^n$ to A' by $\mathbf{P}{\downarrow_{A'}}$ with $a_k \, P_i{\downarrow_{A'}} \, a_l :\Leftrightarrow [a_k P_i a_l$ and $a_k, a_l \in A']$. For instance, for $\mathbf{P} = (a_1a_2a_3, a_3a_1a_2, a_2a_3a_1)$ and $A' = \{a_1, a_3\}$ we have $\mathbf{P}{\downarrow_{A'}} = (a_1a_3, a_3a_1, a_3a_1)$. Conversely, if $A' \supseteq A$ is a superset of A with $A' \smallsetminus A = \{a_{m+1}, \dots, a_{m'}\}$ we define the *lifting* $\mathbf{P}{\uparrow^{A'}}$ of $\mathbf{P} \in \mathcal{P}(A)^n$ to A' by appending alternatives $a_{m+1}, \dots, a_{m'}$ to each ordering P_i below the lowest-ranked alternative from A. That is, for $\mathbf{P} = (a_1a_2a_3, a_3a_1a_2, a_2a_3a_1)$ and $A' = \{a_1, a_2, a_3, a_4\}$ we have $\mathbf{P}{\uparrow^{A'}} = (a_1a_2a_3a_4, a_3a_1a_2a_4, a_2a_3a_1a_4)$. We let ρ or r refer to whole families of mappings and, for instance, write $\rho(\mathbf{P}) = \rho(\mathbf{P}{\downarrow_{A'}})$ if the same alternative $a^* \in A' \subset A$ happens to win for both A and the smaller set A'.

Now consider $A = \{a_1, \dots, a_m\}$ for $m > 2$ and any $\mathbf{w}, \mathbf{w}' \in \mathbb{N}_0^n$ such that $(r^C, \mathbf{w}) \nsim_m (r^C, \mathbf{w}')$. So there exists $\mathbf{P} \in \mathcal{P}(A)^n$ with $r^C|\mathbf{w}(\mathbf{P}) \neq r^C|\mathbf{w}'(\mathbf{P})$. The $\mathbf{w}$ and $\mathbf{w}'$-weighted versions of the majority relation differ at $\mathbf{P}$: if all pairwise comparisons produced the same winners for weights $\mathbf{w}$ and $\mathbf{w}'$, identical Copeland winners would follow. So a weak victory of some a_k

over some a_l for $\mathbf{w}$ turns into a strict victory of a_l over a_k for $\mathbf{w}'$, i.e.,

$$\sum_{i:\, a_k P_i a_l} w_i \geq \sum_{j:\, a_l P_j a_k} w_j \quad \text{and} \quad \sum_{i:\, a_k P_i a_l} w'_i < \sum_{j:\, a_l P_j a_k} w'_j. \tag{6}$$

Then take $A' = \{a_k, a_l\} \subset A$ where $|A'| = 2$ and projection $\mathbf{P}{\downarrow_{A'}}$. (6) implies

$$\sum_{i:\, a_k\, P_i{\downarrow_{A'}}\, a_l} w_i \geq \sum_{j:\, a_l\, P_j{\downarrow_{A'}}\, a_k} w_j \quad \text{and} \quad \sum_{i:\, a_k\, P_i{\downarrow_{A'}}\, a_l} w'_i < \sum_{j:\, a_l\, P_j{\downarrow_{A'}}\, a_k} w'_j. \tag{7}$$

If both inequalities are strict or $k < l$ then $r^C|\mathbf{w}(\mathbf{P}{\downarrow_{A'}}) = a_k \neq r^C|\mathbf{w}'(\mathbf{P}{\downarrow_{A'}}) = a_l$ and hence $(r^C, \mathbf{w}) \nsim_2 (r^C, \mathbf{w}')$. If not, a_l wins also for $\mathbf{w}$ by lexicographic tie breaking but we can consider profile $\mathbf{P}' \in \mathcal{P}(A')^n$ with $a_l P'_i a_k \Leftrightarrow a_k P_i{\downarrow_{A'}}\, a_l$ for all $i \in N$. Then $r^C|\mathbf{w}(\mathbf{P}') = a_l \neq r^C|\mathbf{w}'(\mathbf{P}') = a_k$ and $(r^C, \mathbf{w}) \nsim_2 (r^C, \mathbf{w}')$.

Conversely take $A = \{a_1, a_2\}$ and $\mathbf{w}, \mathbf{w}' \in \mathbb{N}_0^n$ such that $(r^C, \mathbf{w}) \nsim_2 (r^C, \mathbf{w}')$ and $r^C|\mathbf{w}(\mathbf{P}) = a_1 \neq r^C|\mathbf{w}'(\mathbf{P}) = a_2$ for some $\mathbf{P} \in \mathcal{P}(A)^n$. Then

$$\sum_{i:\, a_1 P_i a_2} w_i \geq \sum_{j:\, a_2 P_j a_1} w_j \quad \text{and} \quad \sum_{i:\, a_1 P_i a_2} w'_i < \sum_{j:\, a_2 P_j a_1} w'_j. \tag{8}$$

Consider $A' = \{a_1, a_2, \ldots, a_m\} \supset A$ where $|A'| = m$ and lifting $\mathbf{P}{\uparrow^{A'}}$. (8) implies

$$\sum_{i:\, a_1\, P_i{\uparrow^{A'}} a_2} w_i \geq \sum_{j:\, a_2\, P_j{\uparrow^{A'}} a_1} w_j \quad \text{and} \quad \sum_{i:\, a_1\, P_i{\uparrow^{A'}} a_2} w'_i < \sum_{j:\, a_2\, P_j{\uparrow^{A'}} a_1} w'_j \tag{9}$$

and alternatives $a_3, \ldots, a_m$ lose all weighted majority comparisons against a_1 and a_2 by construction of $\mathbf{P}{\uparrow^{A'}}$. So $r^C|\mathbf{w}(\mathbf{P}{\uparrow^{A'}}) = a_1 \neq r^C|\mathbf{w}'(\mathbf{P}{\uparrow^{A'}}) = a_2$. Hence $(r^C, \mathbf{w}) \nsim_m (r^C, \mathbf{w}')$. In summary, $(r^C, \mathbf{w}) \nsim_2 (r^C, \mathbf{w}') \Leftrightarrow (r^C, \mathbf{w}) \nsim_m (r^C, \mathbf{w}')$ and, a fortiori, $(r^C, \mathbf{w}) \sim_2 (r^C, \mathbf{w}') \Leftrightarrow (r^C, \mathbf{w}) \sim_m (r^C, \mathbf{w}')$.

Proof of Proposition 4

Let $m > n$. Consider $A = \{a_1, \ldots, a_m\}$ and any $\mathbf{w}, \mathbf{w}' \in \mathbb{N}_0^n$ such that $(r^P, \mathbf{w}) \nsim_m (r^P, \mathbf{w}')$. So there exists $\mathbf{P} \in \mathcal{P}(A)^n$ with $r^P|\mathbf{w}(\mathbf{P}) = a_k \neq r^P|\mathbf{w}'(\mathbf{P}) = a_l$. For this $\mathbf{P}$ let

$$\hat{A} := \left\{a \mid \exists i \in N\colon \forall a' \neq a\colon a P_i a'\right\} \tag{10}$$

denote the set of all alternatives that are top-ranked by some voter. (Obviously, $a_k, a_l \in \hat{A}$.) Now define $A' \subset A$ as the union of $\hat{A}$ and some arbitrary elements of $A \setminus \hat{A}$ such that $|A'| = n$. By construction, each $a \in A'$ has the same weighted number of top positions for projection $\mathbf{P}{\downarrow_{A'}}$ as it had for $\mathbf{P}$. So $r^P|\mathbf{w}(\mathbf{P}{\downarrow_{A'}}) = a_k \neq r^P|\mathbf{w}'(\mathbf{P}{\downarrow_{A'}}) = a_l$. Hence $(r^P, \mathbf{w}) \nsim_n (r^P, \mathbf{w}')$.

Analogously, consider $A = \{a_1, \ldots, a_n\}$ and $\mathbf{w}, \mathbf{w'} \in \mathbb{N}_0^n$ such that $(r^P, \mathbf{w}) \nsim_n (r^P, \mathbf{w'})$. A profile $\mathbf{P} \in \mathcal{P}(A)^n$ with $r^P|\mathbf{w}(\mathbf{P}) = a_k \neq r^P|\mathbf{w'}(\mathbf{P}) = a_l$ can then be lifted to $A' = A \cup \{a_{n+1}, \ldots, a_m\}$. By construction, $r^P|\mathbf{w}(\mathbf{P}{\uparrow}^{A'}) = a_k \neq r^P|\mathbf{w'}(\mathbf{P}{\uparrow}^{A'}) = a_l$. Hence $(r^P, \mathbf{w}) \nsim_m (r^P, \mathbf{w'})$. Overall, we can conclude $(r^P, \mathbf{w}) \sim_m (r^P, \mathbf{w'}) \Leftrightarrow (r^P, \mathbf{w}) \sim_n (r^P, \mathbf{w'})$.

Proof of Proposition 5

Let $k > j$ for otherwise arbitrary $j, k \in \{1, \ldots, m\}$. Consider $A = \{a_1, \ldots, a_m\}$ and any profile $\mathbf{P} \in \mathcal{P}(A)^n$ such that player 1 prefers a_2 most and ranks all remaining alternatives lexicographically while player 2 ranks a_2 in k-th position and otherwise agrees with player 1, i.e., suppose $a_2\, P_1\, a_1\, P_1\, a_3\, P_1\, a_4\, \ldots\, a_m$ and $a_1\, P_2\, a_3\, P_2\, a_4\, \ldots\, a_k\, P_2\, a_2\, P_2\, a_{k+1}\, P_2\, a_{k+2}\, \ldots\, a_m$.

The Borda score $j \cdot (m-2) + (m-1)$ of a_1 under $\tilde{\mathbf{w}}_\mathbf{j}$ is at least as big as the corresponding score $j \cdot (m-1) + (m-k)$ of a_2. Since scores of $a_3, \ldots, a_m$ are all strictly smaller than that of a_1, we have $r^B|\tilde{\mathbf{w}}_\mathbf{j}(P) = a_1$. With $\tilde{\mathbf{w}}_\mathbf{k}$, by contrast, a_1's weighted score $k \cdot (m-2) + (m-1)$ is strictly smaller than a_2's corresponding score $k \cdot (m-1) + (m-k)$. Scores of $a_3, \ldots, a_m$ remain smaller than a_1's. So $r^B|\tilde{\mathbf{w}}_\mathbf{k}(P) = a_2$. Hence $(r^B, \tilde{\mathbf{w}}_\mathbf{j}) \nsim_m (r^B, \tilde{\mathbf{w}}_\mathbf{k})$.

Proof of Proposition 6

The claim is obvious for $n = 1$, as each non-degenerate weight then is equivalent to $w_1 = 1$. So consider $m \geq n+1$ for $n \geq 2$. Let $A = \{a_1, \ldots, a_m\}$ and $\mathbf{P^i} \in \mathcal{P}(A)^n$ be any preference profile where the first i players rank alternative a_1 last and the remaining $n - i$ players rank alternative a_2 last. Consider any $\bar{\mathbf{w}}_\mathbf{k}$ and $\bar{\mathbf{w}}_\mathbf{l}$ with $k < l$. Then $r^A|\bar{\mathbf{w}}_\mathbf{k}(\mathbf{P^k}) = a_2 \neq r^A|\bar{\mathbf{w}}_\mathbf{l}(\mathbf{P^k}) = a_3$. So $\mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\mathbf{1},m}, \mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\mathbf{2},m}, \ldots, \mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\mathbf{n},m}$ all differ.

Now assume some $\mathbf{w} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}$ with $w_1 \geq w_2 \geq \ldots \geq w_n$ satisfies $(r^A, \mathbf{w}) \nsim_m (r^A, \bar{\mathbf{w}}_\mathbf{k})$ for all $k \in \{1, \ldots, n\}$. Let l denote the index such that $w_l > 0$ and $w_{l+1} = 0$. Then both $r^A|\mathbf{w}(\mathbf{P})$ and $r^A|\bar{\mathbf{w}}_\mathbf{l}(\mathbf{P})$ equal the lexicographically minimal element in set

$$Z^l(\mathbf{P}) := \Big\{ a \in A \mid \forall i \in \{1, \ldots, l\} \colon \exists a' \in A \colon a P_i a' \Big\} \tag{11}$$

that collects all alternatives not ranked last by any of the players who have positive weight. These coincide for $\mathbf{w}$ and $\bar{\mathbf{w}}_\mathbf{l}$; and $Z^l(\mathbf{P})$ is non-empty because $m \geq n+1$. This holds for arbitrary $\mathbf{P} \in \mathcal{P}(A)^n$. Hence $r^A|\mathbf{w} \equiv r^A|\bar{\mathbf{w}}_\mathbf{l}$, contradicting the assumption that $(r^A, \mathbf{w}) \nsim_m (r^A, \bar{\mathbf{w}}_\mathbf{k})$ for all $k \in \{1, \ldots, n\}$. Consequently, $\mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\mathbf{1},m}, \mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\mathbf{2},m}, \ldots, \mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\mathbf{n},m}$ are all antiplurality classes that exist for $m \geq n+1$ (plus the degenerate $\mathcal{E}_{\mathbf{0},m}$).

– Possibly for online publication –
Appendix B: Minimal representations of committees

n, m	Minimal $\bar{\mathbf{w}}$ for all antiplurality classes $\mathcal{E}^{r^A}_{\bar{\mathbf{w}},m}$							
$3, 3$	1.	(1,0,0)	3.	(1,1,1)	5.	(2,2,1)		
	2.	(1,1,0)	4.	(2,1,1)				
$3, m \geq 4$	1.	(1,0,0)	2.	(1,1,0)	3.	(1,1,1)		
$4, 3$	1.	(1,0,0,0)	6.	(2,1,1,1)	11.	(3,2,2,1)	16.	(4,3,2,2)
	2.	(1,1,0,0)	7.	(2,2,1,0)	12.	(3,3,1,1)	17.	(4,4,2,1)
	3.	(1,1,1,0)	8.	(2,2,1,1)	13.	(3,3,2,1)	18.	(4,4,3,2)
	4.	(1,1,1,1)	9.	(2,2,2,1)	14.	(3,3,2,2)	19.	(5,4,3,2)
	5.	(2,1,1,0)	10.	(3,2,1,1)	15.	(4,3,2,1)		
$4, 4$	1.	(1,0,0,0)	3.	(1,1,1,0)	5.	(2,1,1,1)	7.	(2,2,2,1)
	2.	(1,1,0,0)	4.	(1,1,1,1)	6.	(2,2,1,1)		
$4, m \geq 5$	1.	(1,0,0,0)	2.	(1,1,0,0)	3.	(1,1,1,0)	4.	(1,1,1,1)

Table B-1: Minimal representations of different antiplurality committees

n, m	Minimal $\bar{\mathbf{w}}$ for all Borda classes $\mathcal{E}^{r^B}_{\bar{\mathbf{w}},3}$							
$3, 3$	1.	(1,0,0)	14.	(3,3,2)	27.	(5,4,3)	40.	(8,6,3)
	2.	(1,1,0)	15.	(4,3,1)	28.	(7,4,1)	41.	(9,6,2)
	3.	(1,1,1)	16.	(5,2,1)	29.	(6,5,2)	42.	(8,7,3)
	4.	(2,1,0)	17.	(4,3,2)	30.	(7,5,1)	43.	(8,6,5)
	5.	(2,1,1)	18.	(5,2,2)	31.	(6,5,3)	44.	(10,7,2)
	6.	(2,2,1)	19.	(5,3,1)	32.	(7,5,2)	45.	(11,7,2)
	7.	(3,1,1)	20.	(4,3,3)	33.	(8,5,1)	46.	(9,7,5)
	8.	(3,2,0)	21.	(5,4,1)	34.	(6,5,4)	47.	(10,8,3)
	9.	(3,2,1)	22.	(6,3,1)	35.	(7,5,3)	48.	(11,8,2)
	10.	(4,1,1)	23.	(5,3,3)	36.	(7,6,2)	49.	(11,9,3)
	11.	(3,2,2)	24.	(5,4,2)	37.	(8,5,2)	50.	(13,8,2)
	12.	(3,3,1)	25.	(6,4,1)	38.	(7,5,4)	51.	(12,9,7)
	13.	(4,2,1)	26.	(7,2,2)	39.	(7,6,4)		

Table B-2: Minimal representations of different Borda committees

n	Minimal $\bar{\mathbf{w}}$ for all Copeland classes $\mathcal{E}^{r^C}_{\bar{\mathbf{w}},m}$ and for all classes $\mathcal{E}^{r}_{\bar{\mathbf{w}},2}$ when $r \in \left\{r^A, r^B, r^P\right\}$ and for all weighted voting games $[q;\mathbf{w}]$ with $q = 0.5\sum w_i$			
3	1. (1,0,0)	2. (1,1,0)	3. (1,1,1)	4. (2,1,1)
4	1. (1,0,0,0)	4. (1,1,1,1)	7. (2,2,1,1)	
	2. (1,1,0,0)	5. (2,1,1,0)	8. (3,1,1,1)	
	3. (1,1,1,0)	6. (2,1,1,1)	9. (3,2,2,1)	
5	1. (1,0,0,0,0)	8. (2,1,1,1,1)	15. (3,2,2,1,0)	22. (4,3,2,2,1)
	2. (1,1,0,0,0)	9. (2,2,1,1,0)	16. (4,1,1,1,1)	23. (4,3,3,1,1)
	3. (1,1,1,0,0)	10. (3,1,1,1,0)	17. (3,2,2,1,1)	24. (5,2,2,2,1)
	4. (1,1,1,1,0)	11. (2,2,1,1,1)	18. (3,2,2,2,1)	25. (4,3,3,2,2)
	5. (2,1,1,0,0)	12. (3,1,1,1,1)	19. (3,3,2,1,1)	26. (5,3,3,2,1)
	6. (1,1,1,1,1)	13. (2,2,2,1,1)	20. (4,2,2,1,1)	27. (5,4,3,2,2)
	7. (2,1,1,1,0)	14. (3,2,1,1,1)	21. (3,3,2,2,2)	
6	see next page . . .			

5	. . . see previous page							
6	1.	(1,0,0,0,0,0)	36.	(3,2,2,2,2,1)	71.	(5,4,3,2,1,1)	106.	(5,5,4,3,3,2)
	2.	(1,1,0,0,0,0)	37.	(3,3,2,2,1,1)	72.	(5,4,3,2,2,0)	107.	(6,4,4,3,3,2)
	3.	(1,1,1,0,0,0)	38.	(3,3,2,2,2,0)	73.	(5,4,4,1,1,1)	108.	(6,5,4,3,2,2)
	4.	(1,1,1,1,0,0)	39.	(3,3,3,1,1,1)	74.	(6,3,2,2,2,1)	109.	(6,5,4,3,3,1)
	5.	(2,1,1,0,0,0)	40.	(4,2,2,2,1,1)	75.	(6,3,3,2,1,1)	110.	(6,5,5,2,2,2)
	6.	(1,1,1,1,1,0)	41.	(4,3,2,1,1,1)	76.	(7,2,2,2,2,1)	111.	(7,4,4,3,2,2)
	7.	(2,1,1,1,0,0)	42.	(4,3,2,2,1,0)	77.	(5,4,3,2,2,1)	112.	(7,5,3,3,2,2)
	8.	(1,1,1,1,1,1)	43.	(4,3,3,1,1,0)	78.	(4,4,3,3,2,2)	113.	(7,5,4,3,2,1)
	9.	(2,1,1,1,1,0)	44.	(5,2,2,1,1,1)	79.	(4,4,3,3,3,1)	114.	(7,5,5,2,2,1)
	10.	(2,2,1,1,0,0)	45.	(5,2,2,2,1,0)	80.	(5,3,3,3,2,2)	115.	(8,4,3,3,2,2)
	11.	(3,1,1,1,0,0)	46.	(3,3,2,2,2,1)	81.	(5,4,3,2,2,2)	116.	(6,5,4,4,3,2)
	12.	(2,1,1,1,1,1)	47.	(4,3,2,2,1,1)	82.	(5,4,3,3,2,1)	117.	(6,5,5,3,3,2)
	13.	(2,2,1,1,1,0)	48.	(4,3,3,1,1,1)	83.	(5,4,4,2,2,1)	118.	(7,5,4,3,3,2)
	14.	(3,1,1,1,1,0)	49.	(5,2,2,2,1,1)	84.	(5,5,3,2,2,1)	119.	(7,5,4,4,2,2)
	15.	(2,2,1,1,1,1)	50.	(3,3,2,2,2,2)	85.	(6,3,3,2,2,2)	120.	(7,5,5,3,3,1)
	16.	(2,2,2,1,1,0)	51.	(3,3,3,2,2,1)	86.	(6,4,3,2,2,1)	121.	(7,6,4,3,2,2)
	17.	(3,1,1,1,1,1)	52.	(4,3,2,2,2,1)	87.	(6,4,3,3,1,1)	122.	(7,6,4,3,3,1)
	18.	(3,2,1,1,1,0)	53.	(4,3,3,2,1,1)	88.	(6,4,4,2,1,1)	123.	(7,6,5,2,2,2)
	19.	(3,2,2,1,0,0)	54.	(4,3,3,2,2,0)	89.	(7,3,3,2,2,1)	124.	(8,5,4,3,2,2)
	20.	(4,1,1,1,1,0)	55.	(4,4,2,2,1,1)	90.	(7,3,3,3,1,1)	125.	(8,5,5,3,2,1)
	21.	(2,2,2,1,1,1)	56.	(4,4,3,1,1,1)	91.	(5,4,3,3,3,2)	126.	(9,4,4,3,2,2)
	22.	(3,2,1,1,1,1)	57.	(5,2,2,2,2,1)	92.	(5,4,4,3,2,2)	127.	(7,5,5,4,3,2)
	23.	(3,2,2,1,1,0)	58.	(5,3,2,2,1,1)	93.	(5,4,4,3,3,1)	128.	(7,6,5,3,3,2)
	24.	(4,1,1,1,1,1)	59.	(5,3,3,1,1,1)	94.	(5,5,3,3,3,1)	129.	(8,5,5,4,2,2)
	25.	(2,2,2,2,1,1)	60.	(5,3,3,2,1,0)	95.	(5,5,4,2,2,2)	130.	(8,6,4,3,3,2)
	26.	(3,2,2,1,1,1)	61.	(6,2,2,2,1,1)	96.	(6,4,3,3,2,2)	131.	(8,6,5,3,3,1)
	27.	(3,2,2,2,1,0)	62.	(4,3,3,2,2,1)	97.	(6,4,4,3,2,1)	132.	(9,5,5,3,2,2)
	28.	(3,3,1,1,1,1)	63.	(5,3,3,2,1,1)	98.	(6,5,3,2,2,2)	133.	(7,6,5,4,4,2)
	29.	(3,3,2,1,1,0)	64.	(4,3,3,2,2,2)	99.	(6,5,3,3,2,1)	134.	(8,6,5,4,3,2)
	30.	(4,2,1,1,1,1)	65.	(4,3,3,3,2,1)	100.	(6,5,4,2,2,1)	135.	(8,7,5,3,3,2)
	31.	(4,2,2,1,1,0)	66.	(4,4,3,2,2,1)	101.	(7,3,3,3,2,2)	136.	(9,6,5,4,2,2)
	32.	(5,1,1,1,1,1)	67.	(5,3,2,2,2,2)	102.	(7,4,3,2,2,2)	137.	(9,7,5,4,3,2)
	33.	(3,2,2,2,1,1)	68.	(5,3,3,2,2,1)	103.	(7,4,4,2,2,1)	138.	(9,7,6,4,4,2)
	34.	(3,3,2,1,1,1)	69.	(5,3,3,3,1,1)	104.	(7,4,4,3,1,1)		
	35.	(4,2,2,1,1,1)	70.	(5,4,2,2,2,1)	105.	(8,3,3,3,2,1)		

Table B-3: Minimal representation of different Copeland committees for $m \geq 2$, and of different antiplurality, Borda and plurality committees for $m = 2$, and of different weighted voting games with a simple majority

n,m	Minimal $\bar{\mathbf{w}}$ for all plurality classes $\mathcal{E}^{r^P}_{\bar{\mathbf{w}},m}$							
$3, m \geq 3$	1.	(1,0,0)	3.	(1,1,1)	5.	(2,2,1)		
	2.	(1,1,0)	4.	(2,1,1)	6.	(3,2,2)		
$4, 3$	1.	(1,0,0,0)	10.	(2,2,2,1)	19.	(4,3,2,1)	28.	(5,4,3,1)
	2.	(1,1,0,0)	11.	(3,2,1,1)	20.	(4,3,2,2)	29.	(5,4,3,2)
	3.	(1,1,1,0)	12.	(3,2,2,0)	21.	(4,3,3,1)	30.	(6,4,3,2)
	4.	(1,1,1,1)	13.	(3,2,2,1)	22.	(4,4,2,1)	31.	(6,5,3,2)
	5.	(2,1,1,0)	14.	(3,3,1,1)	23.	(5,2,2,2)	32.	(6,5,4,2)
	6.	(2,1,1,1)	15.	(3,2,2,2)	24.	(4,3,3,2)	33.	(7,4,4,2)
	7.	(2,2,1,0)	16.	(3,3,2,1)	25.	(5,3,3,1)	34.	(7,6,4,2)
	8.	(2,2,1,1)	17.	(4,2,2,1)	26.	(5,3,3,2)		
	9.	(3,1,1,1)	18.	(3,3,2,2)	27.	(5,4,2,2)		
$4, m \geq 4$	1.	(1,0,0,0)	10.	(2,2,2,1)	19.	(4,3,2,1)	28.	(5,4,2,2)
	2.	(1,1,0,0)	11.	(3,2,1,1)	20.	(4,3,2,2)	29.	(5,4,3,1)
	3.	(1,1,1,0)	12.	(3,2,2,0)	21.	(4,3,3,1)	30.	(5,4,3,2)
	4.	(1,1,1,1)	13.	(3,2,2,1)	22.	(4,4,2,1)	31.	(5,4,4,2)
	5.	(2,1,1,0)	14.	(3,3,1,1)	23.	(5,2,2,2)	32.	(6,4,3,2)
	6.	(2,1,1,1)	15.	(3,2,2,2)	24.	(4,3,3,2)	33.	(6,5,3,2)
	7.	(2,2,1,0)	16.	(3,3,2,1)	25.	(5,3,3,1)	34.	(6,5,4,2)
	8.	(2,2,1,1)	17.	(4,2,2,1)	26.	(4,4,3,2)	35.	(7,4,4,2)
	9.	(3,1,1,1)	18.	(3,3,2,2)	27.	(5,3,3,2)	36.	(7,6,4,2)

Table B-4: Minimal representations of different plurality committees

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