On the Democratic Weights of Nations

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January 8, 2015

Abstract

Which voting weights ought to be allocated to single delegates of differently sized groups from a democratic fairness perspective? We operationalize the 'one person, one vote' principle by demanding every individual's influence on collective decisions to be equal a priori. The analysis differs from previous ones by considering intervals of alternatives. New reasons lead to an old conclusion: weights should be proportional to the square root of constituency sizes if voter preferences are independent and identically distributed. This is knife-edged, however, in that preference polarization along constituency lines quickly calls for a Shapley value-based variation of simple proportionality.

Keywords: institutional design; two-tier voting; collective choice; Shapley value; pivot probability; equal representation; random order values

JEL codes: D02; D63; D70; H77

Napel gratefully acknowledges financial support from the Academy of Finland through the Public Choice Research Centre, Turku. We are grateful to the editor in charge, Harald Uhlig, and three anonymous reviewers who suggested several significant improvements to the paper. We also thank Matthew Braham, Manfred Holler, Ulrich Kamecke, Mario Larch, Annick Laruelle, Ines Lindner, Vincent Merlin, Dilip Mookherjee, Abraham Neyman, Andreas Nohn and Larry Samuelson for helpful comments on earlier drafts. We have additionally benefitted from discussions with Robert Hable, Peter Ruckdeschel and feedback on seminar presentations in Aachen, Amsterdam, Augsburg, Bamberg, Bayreuth, Bilbao, Bozen, Caen, Freiburg, Hamburg, Jena, Maastricht, Munich, Paderborn, Paris, Salerno, Siegen, Tilburg, Tübingen, Turku, Verona, Zurich, at the Public Choice Society's 50th anniversary conference, and the Economic Theory committee of the Verein für Socialpolitik. The usual caveat applies.

I. Introduction

In decision-making bodies with a divisional or regional structure, the sizes or voting weights of different delegations commonly vary in the numbers of represented constituents. How they do so varies, too. In the US Electoral College, for instance, each state has two votes in addition to a number which is proportional to population size. California and Wyoming with around 11.9% and 0.2% of the US population thus end up holding around 10.2% and 0.6% of votes on the US President. In contrast, the most and least populous member states of the EU – Germany and Malta – currently have about 8.2% and 0.9% of votes in the Council of the European Union but comprise 15.9% and 0.1% of the EU population; the respective mapping from population size to voting weight is, very roughly, a square root function.[1] Delegates in other decision-making bodies, such as the Senate of Canada, the Assembly of the African Union, the Governing Council of the European Central Bank and many a university senate or council of a multi-branch NGO, have voting weights that are yet more concave in the number of represented individuals, or even flat.

This paper analyzes democratic fairness of different voting weight arrangements. Individuals choose delegates in disjoint constituencies (bottom tier) and these representatives take collective decisions in an assembly (top tier). We investigate a practical question: which mapping – possibly linear, possibly strictly concave or constant – should determine the top-tier voting weights of delegates from differently sized constituencies? Barberà and Jackson (2006) and Koriyama et al. (2013), among others, have studied this question for binary policy spaces, with the objective to maximize a utilitarian welfare function. We focus on the basic democratic principle of '*one person, one vote*'. Our corresponding conception of equitable institutional design is that all bottom-tier voters should wield *equal influence* on collective decisions – at least under stylized ideal conditions behind a constitutional veil of ignorance.

The major difference to the existing literature is that agents face an interval of policy alternatives, rather than binary ones. This opens up the analysis to a collection of economic issues (such as tax rates, monetary policy, spending on climate change mitigation) that otherwise would not be covered. We assume that voter preferences are single-peaked. Bargaining, political competition and other types of interaction at the constituency level can then be captured in reduced form by considering the respective *median voter*. Specifically, the realized median preference of each constituency is presumed to act as its representative. The representatives then use a weighted voting rule with a 50% majority threshold and adopt the *Condorcet winner* among their ideal points, i.e., the policy which beats all alternatives in a pair-wise vote. This coincides,

[1] A least squares power-law regression of EU Council voting weights w_i on population sizes n_i results in $w_i = c \cdot n_i^{0.48}$ with $R^2 \approx 0.95$. The current Council voting rules involve two other but essentially negligible criteria, and will be changed in 2017 into a more proportional system.

formally, with the weighted median in the assembly.

The distributions of voter preferences and the delegates' voting weights jointly determine how often a given constituency is pivotal – that is, how often its most preferred policy is the assembly's weighted median. Our design objective calls for weights to be such that each constituency's probability of being pivotal is *proportional to its population size*, making the a priori assumption that voter preferences are all identically distributed as well as mutually independent across constituencies.

Proportionality of pivot probabilities at the top tier is linked to equal individual influence in two ways. The first is to view an individual citizen's influence as his or her chances to induce a collective outcome in accordance with the personal preference's ideal point: that is, to be a median voter of the decisive constituency. The probability of being the local median is inversely proportional to the constituency's population size. If this is balanced by proportional pivot probabilities for the representatives, the democratic playing field is level.

Alternatively, one can identify a constituent's influence with the anticipated effect of taking part in the decision making process. Even if the constituency's preferences coincide with those of only one individual, *every* constituency member affects who this is. If, say, some left-wing voter dropped out – so an ideal point to the left of the median is deleted from the constituency's preference distribution – then the median's location would shift to the right. The expected size of such shift and hence the influence associated with democratic participation are inversely proportional to constituency size. Again, proportionality is required in order to avoid bias.

The relation of *heterogeneity within each constituency* and *heterogeneity across constituencies* turns out to be the critical parameter for a fair weight allocation, because it determines how the probability distributions of the constituency medians vary with population sizes. A greater number of preference draws generally reduces the variance of the resulting median. The representative of a large constituency will therefore frequently hold a central political position, at least if all preferences are independent. This raises the odds of being the assembly's median compared to delegates from small constituencies, who are more often outliers. In view of this size-related advantage of large constituencies, strictly concave or 'degressively proportional' weights can be enough to induce proportionality of pivot probabilities in the assembly. However, positive correlation within the constituencies – corresponding to local fixed effects and reflecting heterogeneity across constituencies – slows the reaction of variance to local population size. If, consequently, representatives of large constituencies have little or no locational advantage in the assembly, they need more progressive margins for their voting weights.

Square root and linear weighting rules emerge from these considerations in important benchmark cases. The former is advisable if all voter preferences are mutually independent (Corollary 1), while the considered democratic ideal calls for linear allocations when there is sufficient heterogeneity across constituencies, i.e., if voters are polarized along constituency lines (Corollary 2). These conclusions follow from new limit results on committee decisions on intervals which may also find other applications (Theorems 1 and 2).

Of course, an individual's probability of being pivotal and also the expected policy effects of participation will be very small for most real-life population figures.[2] In relative terms, however, these probabilities can differ widely across constituencies when weights are chosen arbitrarily. They should not, first, as a matter of democratic principle. Second, though we make no cardinal assumptions in the analysis, large issues can be at stake. The associated variation of expected utilities may hence be significant. A compelling explanation of why people vote at all is that they care about the general good, i.e., they have social preferences. If election of candidate X rather than Y, or a policy shift to the left or right, would raise quality of life by the equivalent of \$100 for ten million fellow citizens in the eyes of a given, socially-minded individual, then tiny prospects of affecting the outcome become billion-dollar lotteries, whose allocation matters (see Edlin et al. 2007). Finally, variation of pivot probabilities makes it profitable for politicians to concentrate their attention and resources, i.e., target policies to states, districts and voter groups who have a high chance of being pivotal. A considerable empirical literature documents how inequality in political influence produces inequality in public expenditures.[3] There is also evidence that voters who believe their participation to be pivotal turn out more likely (see, for example, Duffy and Tavits 2008) and these turnout rates affect policy (compare Mueller and Stratmann 2003 or Fumagalli and Narciso 2012).

The weighting implications of ensuring an equal say for all voters in a two-tier system under normatively motivated a priori assumptions were first formally considered by Lionel S. Penrose in 1946.[4] The institutional design of a successor to the League of Nations – today's UNO – was then being discussed. Penrose (1946) argued that the most intuitive solution to the democratic weight allocation problem, i.e., weights proportional to constituency sizes, ignores "elementary statistics of majority

[2]Combining US voter figures with poll data, Gelman et al. (2012) estimated chances for a single vote being decisive in the 2008 presidential elections as about 1 in 60 million, but up to 1 in 10 million in some small and midsize states who were near the national median politically. Persuading 500 people in, e.g., New Mexico to change their votes would have provided a 1 in 6,000 chance of swinging the election ex ante.

[3]For instance, see Atlas et al. (1995) and Knight (2008) on US federal spending, Ansolabehere et al. (2002) on spending by US states, and Rodden (2002) for a study of the EU Council. On a related note, Strömberg (2008) has shown pivot considerations to be an important determinant for the allocation of campaign resources in US presidential elections.

[4]Informal investigations date back to anti-federalist writings by Luther Martin, a delegate from Maryland to the Constitutional Convention in Philadelphia in 1787. See Riker (1986).

voting". Namely, if there are only two policy alternatives ('yes' or 'no') and all individual decisions are statistically *independent and equiprobable* then top-tier voting weights need to be such that the induced pivot probabilities of the delegates are proportional to the square root of represented population sizes. The corresponding suggestion is known as *Penrose's square root rule*.

This rule has provided a benchmark for a long list of applied studies on voting power in the US, EU, or IMF (see Felsenthal and Machover 2004; Grofman and Feld 2005; Fidrmuc et al. 2009; Leech and Leech 2009; Miller 2009, 2012, and the references therein). Politicians and diplomats may not care about Penrose's reasoning as such, but they have invoked his suggestion when it has fitted their interests.[5]

The special roles of square root and linear weight allocations have been confirmed, qualified, and disputed in numerous investigations of two-tier voting systems, both empirically (see Gelman et al. 2002, 2004) and theoretically. Besides the a priori influence of voters (Chamberlain and Rothschild 1981; Felsenthal and Machover 1998; Laruelle and Valenciano 2008b; Kaniovski 2008), utilitarian welfare maximization has played a particularly prominent role (e.g., Barberà and Jackson 2006; Beisbart and Bovens 2007; Laruelle and Valenciano 2008b; Koriyama et al. 2013). Moreover, avoidance of majoritarian paradoxes such as in the US presidential elections of 2000 has featured as desirable ideal (Felsenthal and Machover 1999; Kirsch 2007; Feix et al. 2008).

This literature has made several departures from Penrose's original assumptions but has focused almost entirely on *binary* decisions.[6] Dichotomous options provide no scope for negotiation and bargaining. This may be suited to political decisions between just two exogenously available candidates, or perhaps on whether there is to be taxation, regulation, or aid. But it does not fit competition between many policies and economic decisions on *levels*, such as rates of taxation or their progressiveness, the intensity or scope of regulation, aid's scale or means test requirements. We therefore analyze the equitable choice of voting weights for a richer set of alternatives. Though the relevant statistics are entirely different (and no longer so elementary) the design implications differ, to our own surprise, only little. Our analysis reinforces and corroborates an increasingly robust pattern in the literature: as originally argued by Penrose (1946), ex ante independent and identical voters require weight allocations based on the square roots of population sizes. But sufficient dissimilarity between

[5]A particularly notorious case involved the then Polish president and prime minister in the EU's negotiations of the Treaty of Lisbon. See, e.g., *The Economist* (2007, June 14th).

[6]We are aware of the following exceptions only: Laruelle and Valenciano (2008a) suggest a "neutral" top-tier voting rule when policy alternatives give rise to a Nash bargaining problem. Le Breton et al. (2012) investigate fair voting weights in case of the division of a transferable surplus, i.e., for a simplex of policy alternatives. Maaser and Napel (2007, 2012, 2014) report Monte Carlo simulation results for influence-based, majoritarian, and welfarist objective functions in the median voter environment which we will here investigate analytically.

constituencies renders most people's intuition correct – 'one person, one vote' calls for plain proportionality.

The remainder of the paper is organized as follows. In Section II, we spell out our stylized model of two-tier decision making and formalize the institutional design problem. The main result for simple majority rule and a large number of constituencies is presented in Section III, together with its voting weight implications in case all preference draws are independent. We explore different majority requirements and the effect of adding heterogeneity across constituencies to that within in Section IV. We conclude in Section V and provide proofs in a mathematical appendix.

II. Model and Design Problem

A. Agents and Preferences

Consider a partition $\mathfrak{C}^m = \{C_1, \ldots, C_m\}$ of a large number n of voters into $m < n$ disjoint *constituencies* with $n_i = |C_i| > 0$ members each. The preferences of any voter $l \in \{1, \ldots, n\} = \bigcup_i C_i$ are assumed to be single-peaked over a convex one-dimensional *policy space* $X \subseteq \mathbb{R}$, i.e., a finite or infinite real interval. Voter l's *ideal point* ν^l is conceived of as the realization of a continuous random variable. A given profile $(\nu^1, \ldots, \nu^n)$ of ideal points could reflect voter preferences in an abstract left–right spectrum or regarding a specific one-dimensional variable such as the location or scale of a public good, an exemption threshold for regulation, a transfer level, etc.

We assume throughout our analysis that voter ideal points are a priori *identically distributed*. Moreover, it is assumed that ideal points are mutually *independent across constituencies*. We do, however, allow for a specific form of ideal points being correlated within each constituency.

In particular, we conceive of the ideal point ν^l of any voter $l \in C_i$ as the sum

$$\nu^l = \mu_i + \epsilon^l \tag{1}$$

of a constituency-specific shock μ_i which has distribution H and a voter-specific shock ϵ^l with distribution G and continuous density g. Variables $\epsilon^1, \ldots, \epsilon^n$ and $\mu_1, \ldots, \mu_m$ are all mutually independent. The variance of G, $\sigma_G^2 < \infty$, can be interpreted as a measure of *heterogeneity within each constituency*, reflecting natural variation of political and economic preferences. If distribution H of μ_i is non-degenerate, it reflects a common attitude component of preferences within each constituency. H's variance $\sigma_H^2 < \infty$ is a straightforward measure of *heterogeneity across constituencies*: even though it is assumed that preferences in all constituencies vary between left–right, high tax–low tax, etc. in a similar manner, the locations of the respective ranges of opinion can differ between constituencies from an interim perspective.

A priori, as G and H are the same for all voters l and constituencies C_i, ideal points are distributed identically. The *i.i.d. case* of $\sigma_H^2 = 0$ in which they are also independent is an important benchmark. However, constituencies differ in nothing but size a priori also if $\sigma_H^2 > 0$: ν^l and ν^k are correlated with the same coefficient $\sigma_H^2/(\sigma_H^2 + \sigma_G^2)$ whenever $l, k \in C_i$, for every constituency $C_i \in \mathfrak{C}^m$.

B. Two-Tier Median Voter Model

A collective decision $x^* \in X$ on the issue at hand is taken by an assembly of representatives $\mathcal{R}^m$ which consists of one representative from each of the m constituencies. Without committing to any particular procedure for within-constituency preference aggregation – such as bargaining, electoral competition, or a central mechanism – it will be assumed that preferences of C_i's representative coincide with those of its respective median voter, i.e., the location of the ideal point of representative i is

$$\lambda_i \equiv \text{median}\,\{\nu^l : l \in C_i\}. \tag{2}$$

This leaves aside agency problems and other reasons for why the preferences of a constituency's representative might not be congruent or at least sensitive to its median voter.[7]

In the top-tier assembly $\mathcal{R}^m$, constituency C_i has voting weight $w_i \geq 0$. Any coalition $S \subseteq \{1, \ldots, m\}$ of representatives which achieves a combined weight $\sum_{j \in S} w_j$ above

$$q^m \equiv \frac{1}{2} \sum_{j=1}^{m} w_j, \tag{3}$$

i.e., which has a *simple majority* of total weight, is winning and can pass proposals to implement some policy $x \in X$. This voting rule is denoted by $[q^m; w_1, \ldots, w_m]$.

Now consider the random permutation of $\{1, \ldots, m\}$ that makes $\lambda_{k:m}$ the k-th leftmost ideal point among the representatives for any realization of $\lambda_1, \ldots, \lambda_m$. That is, $\lambda_{k:m}$ is their k-th order statistic. We will disregard the zero probability events of two or more constituencies having identical ideal points and define the random variable P by

$$P \equiv \min \Big\{ j \in \{1, \ldots, m\} : \sum_{k=1}^{j} w_{k:m} > q^m \Big\}. \tag{4}$$

The ideal point $\lambda_{P:m}$ of representative $P:m$ cannot be beaten by any alternative $x \in X$ in a pairwise vote, i.e., it is in the *core* of the voting game defined by ideal points

[7] See, e.g., Gerber and Lewis (2004) for empirical evidence on how district median voters and partisan pressures jointly explain legislator preferences, and for a short discussion of the related theoretical literature. We remark that Theorems 1 and 2 below will *not* require identity (2) to hold.

$\lambda_1, \ldots, \lambda_m$, weights $w_1, \ldots, w_m$ and quota q^m. We assume that the policy x^* agreed by $\mathcal{R}^m$ lies in the core. Whenever that is single-valued, $\lambda_{P:m}$ actually beats every other alternative $x \in X$ and is the so-called *Condorcet winner* in $\mathcal{R}^m$. In order to avoid inessential case distinctions, we assume that $\mathcal{R}^m$ agrees on $\lambda_{P:m}$ also in the non-generic and knife-edge cases of the entire interval $[\lambda_{P-1:m}, \lambda_{P:m}]$ being majority-undominated, i.e., the collective choice equals[8]

$$x^* \equiv \lambda_{P:m}. \tag{5}$$

Representative $P\colon m$ will be referred to as either the *pivotal representative* or the *weighted median* of $\mathcal{R}^m$. Banks and Duggan (2000) and Cho and Duggan (2009) provide equilibrium analysis of non-cooperative bargaining which supports policy outcomes inside or close to the core. Note that for x^* determined in this way, no constituency's median voter has an incentive to choose a representative whose ideal point differs from her own one, that is, to misrepresent her preferences (cf. Moulin 1980; Nehring and Puppe 2007).

C. *Influence and Equal Democratic Representation*

Individuals differ only with respect to the sizes of their constituencies a priori; hence the voting weights which are allocated to their representatives should not create a disadvantage for members of large constituencies, nor for those of any other size. Our corresponding objective is to implement the influence aspect of the 'one person, one vote' principle. More precisely, given a partition $\mathfrak{C}^m = \{C_1, \ldots, C_m\}$ of n voters into constituencies and distributions G and H which describe heterogeneity of individual preferences within and across constituencies, we would like to allocate voting weights $w_1, \ldots, w_m$ such that each voter a priori has *equal influence on the collective decision* $x^* \in X$.

There are two complementing ways of operationalizing a voter's influence on x^*. They extend the classical approach in the analysis of binary elections, where an individual l is influential if the election is tied without l's vote (or one vote away from a tie). In that case, voter l is decisive in two senses: (1) the election outcome coincides with l's vote and is *sensitive* to it, i.e., a change in l's vote would change the outcome; (2) given the decisions of all others, individual l's *turnout* matters, i.e., it makes a difference to the outcome whether l votes or not (at least with a positive probability which depends on tie-breaking). Direct analogues in our continuous world are that (1′) x^* coincides (approximately) with l's ideal point and idiosyncratic shifts of ν^l translate into shifts of x^*, i.e., $\partial x^*/\partial \nu^l > 0$; and (2′) whether l expresses her preferences or not, i.e., whether ideal point ν^l is incorporated into the local median, affects x^*'s location.

In contrast to the binary world, influence in the sensitivity and in the turnout

[8] A sufficient condition for the core to be single-valued is that the vector of weights satisfies $\sum_{j \in S} w_j \neq q^m$ for each $S \subseteq \{1, \ldots, m\}$.

senses are associated with different events. That voter $l \in C_i$ is influential in the sensitivity sense requires that *l is C_i's median voter* for an odd population size n_i. By our assumptions – ideal points v^l and v^k are identically distributed and at least conditionally independent if $l, k \in C_i$ – the probability of l being the local median is $1/n_i$, i.e., *inversely proportional* to constituency size. Conditioning on the intersection of this event and that of C_i's representative being pivotal at the top tier, we have $\partial x^*/\partial v^l = 1$. Since events $\{v^l = \lambda_i\}$ and $\{x^* = \lambda_i\}$ are independent,[9] we can quantify a priori influence of voter $l \in C_i$ as

$$\mathbf{E}\Big[\partial x^*/\partial v^l\Big] = \frac{\pi_i(\mathcal{R}^m)}{n_i} \tag{6}$$

where

$$\pi_i(\mathcal{R}^m) \equiv \Pr(P : m = i) \tag{7}$$

denotes the probability of C_i's representative being the assembly's weighted median. The same follows for an even number n_i.[10] Equal influence on collective decisions hence demands that $\pi_i(\mathcal{R}^m)$ is proportional to n_i.

Influence in the turnout sense does not require l to be a median voter in her constituency because *every* member of C_i affects the constituency's median position by participating. Dropping a voter from sample $\{v^l : l \in C_i\}$ who is to the left of λ_i would give rise to a new median position $\lambda_i' > \lambda_i$ on the right; dropping one on the right shifts λ_i left. (If the median voter herself fails to participate, λ_i is replaced by the midpoint $\lambda_i' \gtrless \lambda_i$ of its neighbors.) The probability of a given individual $l \in C_i$ having *some* influence in this sense – and thus a reason to vote – is $\pi_i(\mathcal{R}^m)$. However, the *extent* of influence (conditional on pivotality of C_i in $\mathcal{R}^m$) depends on constituency size. To see this, suppose all ideal points are independent and distributed uniformly on $[0, 1]$. Then the expected location of the k-th left-most position in C_i is $k/(n_i + 1)$. C_i's expected median position is $\mathbf{E}[\lambda_i] = 1/2$, and for an even n_i would be replaced by $\mathbf{E}[\lambda_i'] = (n_i/2 + 1)/(n_i + 1) > 1/2$ if a left-wing voter dropped out. The shift's expected size is $\mathbf{E}[\lambda_i' - \lambda_i] = 1/[2(n_i + 1)]$, i.e., it is about halved if population is doubled.

The effects of adding (or deleting) an observation to a given sample are rigorously studied in mathematical statistics, in the context of robust estimation. There, the *influence function* of the median functional has been established to be

$$\psi(v^l) = \frac{\operatorname{sign}(v^l - \mathrm{M})}{2f(\mathrm{M})} \tag{8}$$

[9] The first event only entails information about the identity of C_i's median, not its location.

[10] The median position λ_i then is the midpoint of the ideal points of C_i's two middlemost voters. The probability of l being one of them is $2/n_i$, and then $\partial x^*/\partial v^l = \frac{1}{2}$ if C_i's representative is pivotal. This yields $\mathbf{E}[\partial x^*/\partial v^l] = \pi_i(\mathcal{R}^m)/n_i$ again.

where f denotes the density of v^l's distribution function F, and M is F's median. It can be used in order to write

$$\lambda_i = \mathrm{M} + \frac{1}{n_i} \sum_{l \in C_i} \psi(v^l) + R_i \tag{9}$$

with a residual term R_i which vanishes in probability as $n_i \to \infty$.[11] That is, we may – with only small error – conceive of C_i's median position as the result of starting at the common theoretical median M and then doing n_i equidistant jumps of size $1/[2f(\mathrm{M}) \cdot n_i]$ to the right or left depending on whether $v^l > \mathrm{M}$ or $v^l < \mathrm{M}$. The effect of an individual voter l expressing her preferences or not on λ_i is thus inversely proportional to C_i's population size.[12] Providing all voters with the same influence from turning out therefore requires proportionality of probability $\pi_i(\mathcal{R}^m)$ to n_i.

It follows that irrespective of which type of individual influence we seek to equalize across constituencies, our institutional design objective consists of solving the following

Problem of Equal Democratic Representation:
Find a mapping from constituency sizes $n_1, \ldots, n_m$ to voting weights $w_1, \ldots, w_m$ for the representatives in $\mathcal{R}^m$ such that

$$\frac{\pi_i(\mathcal{R}^m)}{\pi_j(\mathcal{R}^m)} \approx \frac{n_i}{n_j} \text{ for all } i, j \in \{1, \ldots, m\}. \tag{10}$$

One might conjecture that we can simply use population sizes as voting weights, assuming w_i translates into $\pi_i(\mathcal{R}^m)$ linearly. This would be too quick on two grounds: the latter presumption can be unwarranted and, importantly, the distributions of representatives' ideal points affect pivot probabilities. The problem's solution will therefore depend on how constituencies' median preferences vary with their sizes, and how they interact with voting weights.[13]

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[11]See, e.g., Van der Vaart (1998, Example 20.5 and Corollary 21.5).

[12]Equation (9) derives from what statisticians call *von Mises calculus* (after the younger brother of the Austrian economist). It mimics Taylor approximation of a real function in the world of statistical functionals. Unfortunately, $R_i \in o_{F^{n_i}}(n_i^{-0.5})$; that is, the remainder term vanishes only at a square root rate in general. This means we cannot, strictly speaking, conclude directly from (9) that the expected size of a shift of λ_i due to dropping v^l is proportional to n_i^{-1}. However, with a bit of effort, one can show that $\lim_{n_i \to \infty} 2(n_i + 1) f(\mathrm{M}) \cdot \Delta_{n_i}(f) = 1$, where $\Delta_{n_i}(f)$ is the expected change of the median caused by deleting one of n_i observations from the sample, given that all $v^l \in C_i$ are conditionally i.i.d. with continuous positive density f at M. A proof is available from the authors.

[13]Re-partitioning the population into constituencies of equal size – i.e., appropriate redistricting – is, of course, a possibility for altogether evading the considered problem. Our analysis is concerned with those cases where historical, geographical, cultural, and other reasons exogenously have defined a partition $\mathfrak{C}^m$ which cannot easily be changed. See Coate and Knight (2007) on socially optimal districting and Gul and Pesendorfer (2010) on strategic issues in redistricting.

D. Discussion

Before we start to investigate the stated design problem formally in Section III, two valid concerns should be addressed. A first objection to the pivotality-based condition (10) could be that a voter's associated indirect influence on outcomes is too small to worry about. We beg to differ because even tiny numbers can matter. In particular, a level democratic playing field may be valued by the constituents as such: influence of a member of C_i should not be systematically larger or smaller than that of a member of C_j even if both are minuscule. Ratios are then more relevant than absolute levels. We would interpret public reaction to suggested re-weightings, for instance in the run-up to the Lisbon Treaty's reform of EU voting rules, along such non-instrumental lines. Moreover, small probabilities or policy shifts can matter also instrumentally. As discussed in detail by Edlin et al. (2007) it is an empirically plausible, rational explanation of *why* people vote that they care about the wider social benefits of policy (e.g., the entire government budget, not only what they personally get out of it and pay). If voters attend rallies or vote on a rainy election day against all odds because they perceive a sufficiently large stake, then pivotality has distributional consequences for their welfare (as well as turnout incentives). Finally, though we derived condition (10) from democratic principles applied to individuals, it relates to the influence of *constituencies*. Chances of being decisive for an ultimate decision are key to a constituency's powers and have non-negligible financial implications for it. They can be required to satisfy proportionality for reasons other than individual citizens' indirect influence.

With these arguments in mind, one might alternatively object that

$$\frac{\pi_i(\mathcal{R}^m)}{\pi_j(\mathcal{R}^m)} = \frac{n_i}{n_j} \text{ for all } i, j \in \{1, \ldots, m\} \tag{11}$$

should actually hold with equality, not just approximately. Unfortunately, due to the discrete nature of weighted voting, condition (11) cannot be satisfied by *any* weight vector $(w_1, \ldots, w_m)$ for most combinations of $\mathfrak{C}^m$, G, and H.[14] So the realistic task is either to minimize distance between $(n_1, \ldots, n_m)/n$ and the probability vector $(\pi_1(\mathcal{R}^m), \ldots, \pi_m(\mathcal{R}^m))$ induced by $w_1, \ldots, w_m$, or to find a way by which condition (11) is satisfied in an asymptotic sense – which corresponds to our notion of holding approximately. We follow the latter approach and, in particular, will not discuss here how one might solve the respective (non-trivial) discrete minimization problem for a specific partition $\mathfrak{C}^m$ and specific distributions G and H. Our ambition is to identify

[14]There are finitely many structurally different weighted voting games for any given m. Their number – related to *Dedekind's problem* in discrete mathematics – and the associated sets of feasible vectors $(\pi_1(\mathcal{R}^m), \ldots, \pi_m(\mathcal{R}^m))$ grow fast in m but include $(n_1, \ldots, n_m)/n$ only in special cases. See Kurz (2012).

weighting rules which satisfy the 'one person, one vote' criterion approximately but generally, that is: they induce $\pi_i(\mathcal{R}^m)/\pi_j(\mathcal{R}^m) \approx n_i/n_j$ for any i, j for a large class of partitions $\mathfrak{C}^m$ and they require only qualitative information on voter heterogeneity.

To conclude this discussion, let us reiterate that the considered median voter model of equal representation in two-tier decision making is an admittedly big simplification. Many collective decisions involve more than just a single dimension in which voter preferences differ. We ignore that voting might involve private information about some state variable (Feddersen and Pesendorfer 1996, 1997; Bouton and Castanheira 2012), and typical agency problems connected to imperfect monitoring and infrequent delegate selections. Empirical evidence highlights that a representative may take positions that differ significantly from his district's median when voter preferences within that district are sufficiently heterogeneous (see, e.g., Gerber and Lewis 2004). Still, we take it that the best intuitions about fairness are captured by simplifying thought experiments of a veil of ignorance kind. The analysis of the described stylized world – no friction, particularly well-behaved preferences which are a priori identical for all – is useful in this way. It shows the limitations of and justifications for the simple intuition that weights should be proportional to the number of represented constituents, in a framework that goes beyond the binary world analyzed by Penrose and most others.

III. Fair Voting Weights for Many Constituencies

We now study how pivot probabilities $\pi_i(\mathcal{R}^m)$ in the assembly depend on voting weights and the ideal point distributions of delegates in general. We will then apply this knowledge to the problem of equal representation.

A. Asymptotic Behavior of Pivot Probabilities

Our perspective in this section is an asymptotic one, as in essentially all related literature since Penrose (1946). Studying the case when the number of constituencies is large has two benefits. It helps with the statistical analysis and, moreover, we avoid normative conundrums that can arise for a small number of constituencies. To see this second point, consider $m = 2$ with constituency C_1 twenty times more populous than C_2 and assume almost perfect preference correlation within constituencies. It is then very debatable whether $w_1 > w_2$ (dictatorship of C_1) or equal weights would be the 'fairer' top-tier voting rule. And the welfare loss, in utilitarian terms at least, of allowing delegate 2 to be pivotal could be enormous. Both unavoidable residual inequality as well as possible conflict between democratic fairness and utilitarian normative ideals decrease quickly as the number of independent voter groups rises.

Unfortunately, very few tangible results exist on the distribution of order statistics like the median from potentially *differently* distributed random variables (our delegate ideal points $\lambda_1, \ldots, \lambda_m$); almost nothing seems to be known about the distribution of a weighted median. It still turns out to be possible to characterize the probability $\pi_i(\mathcal{R}^m)$ of some delegate being the weighted median as $m \to \infty$. To this end, we conceive of $\mathcal{R}^1 \subset \mathcal{R}^2 \subset \mathcal{R}^3 \subset \ldots$ as a chain of assemblies with more and more members.

Any representative $i \in \mathbb{N}$ in the chain is endowed with a voting weight $w_i \geq 0$ and has a random ideal point λ_i with absolutely continuous distribution F_i. Some technical conditions will need to be imposed on weights and F_i's density but we can consider assemblies $\mathcal{R}^m$ with fairly arbitrary weighted voting rules $[q^m; w_1, \ldots, w_m]$ and independent ideal point distributions $F_1, \ldots, F_m$ for now.[15] The obtained characterization, therefore, may have applications that are unrelated to two-tier voting systems.

The considered sequences of weights $w_1, w_2, w_3, \ldots$ and ideal point distributions $F_1, F_2, F_3, \ldots$ are assumed to satisfy a weak form of replica structure. The reason is that otherwise our ratio of interest, $\pi_i(\mathcal{R}^m)/\pi_j(\mathcal{R}^m)$, need not converge even if every delegate's relative voting weight goes to zero.[16] We therefore require, first, that all representatives $i \in \mathbb{N}$ belong to one of an arbitrary but finite number of *representative types* $\theta \in \{1, \ldots, r\}$. All representatives are mutually independent but those of the same type have the same weight and ideal point distribution, i.e., there exists a mapping $\tau \colon \mathbb{N} \to \{1, \ldots, r\}$ such that $\tau(j) = \theta$ implies λ_j has distribution F_θ and $w_j = w_\theta$. If, second, each type θ maintains a *non-vanishing share*

$$\beta_\theta(m) \equiv |\{k \in \{1, \ldots, m\} \colon \tau(k) = \theta\}|/m \tag{12}$$

of representatives in $\mathcal{R}^m$ as $m \to \infty$, we call $\mathcal{R}^1 \subset \mathcal{R}^2 \subset \mathcal{R}^3 \subset \ldots$ a *regular chain*.

Regarding the ideal point distributions $F_1, F_2, F_3 \ldots$ we require that they are *locally well-behaved* near their common median M: each associated density f_i is positive at M and varies at most quadratically, i.e., $f_i(\mathrm{M}) > 0$ and $|f_i(x) - f_i(\mathrm{M})| \leq c(x - \mathrm{M})^2$ for some $c \geq 0$ in a neighborhood of M.[17] With these restrictions, the following is shown to hold in Appendix 1:

[15] A representative's ideal point might also be the average of a few ideal points (e.g., members of a local coalition government or oligarchy) or reflect the interests of a constituency dictator; weights might be unrelated to population sizes. Technically, for fixed $F_1, \ldots, F_m$, $\pi(\mathcal{R}^m)$ amounts to a specific *quasivalue* or *random order value*. See, e.g., Monderer and Samet (2002).

[16] This is illustrated by the sequence $\{\boldsymbol{w}^m\}_{m\in\mathbb{N}}$ with $\boldsymbol{w}^m = (1, 2, \ldots, 2) \in \mathbb{R}^m$. Representative 1 is either a null player with $\pi_1(\mathcal{R}^m) = 0$ or, supposing that the ideal point distributions $F_1, \ldots, F_m$ are identical, $\pi_i(\mathcal{R}^m) = \frac{1}{m}$ for *all* $i = 1, \ldots, m$ depending on whether m is odd or even. So $\pi_1(\mathcal{R}^m)/\pi_2(\mathcal{R}^m)$ alternates between 0 and 1. More complicated examples of non-convergence can be constructed by having $\{\boldsymbol{w}^m\}_{m\in\mathbb{N}}$ oscillate in a suitable fashion.

[17] In the i.i.d. case, local well-behavedness of n_i individual ideal points – implied, e.g., by a symmetric C^2 density – is inherited by their median λ_i. A sufficient condition for well-behavedness in the non-i.i.d. case where ν^l has density $g * h\,(\mathrm{M}) > 0$ is that g and h are symmetric and C^1.

Theorem 1. *Consider a regular chain $\mathcal{R}^1 \subset \mathcal{R}^2 \subset \mathcal{R}^3 \subset \ldots$ of assemblies. If the ideal point distribution F_θ has median* M *with locally well-behaved density f_θ for each representative type $\theta \in \{1, \ldots, r\}$ then*

$$\lim_{m\to\infty} \frac{\pi_i(\mathcal{R}^m)}{\pi_j(\mathcal{R}^m)} = \frac{w_i f_i(\mathrm{M})}{w_j f_j(\mathrm{M})} \tag{13}$$

supposing that $w_j > 0$.

B. Democratic Weights in General

Theorem 1 allows to give a simple general answer to the problem of equal representation when many constituencies are involved. In particular, a comparison of equations (10) and (13) immediately yields that choosing

$$(w_1, \ldots, w_m) \propto \left(\frac{n_1}{f_1(\mathrm{M})}, \ldots, \frac{n_m}{f_m(\mathrm{M})}\right) \tag{14}$$

achieves approximately equal representation. The implicit presumption here is, of course, that the voting weights $w_1, \ldots, w_m$ prescribed by (14) do not give rise to some of the problematic issues which regularity rules out in Theorem 1 (e.g., a single constituency having a majority of weight).

Typically, for a fixed number of constituencies, one can slightly raise equality of representation relative to suggestion (14) by letting a *voting power index* capture some of the combinatorial aspects of voting, which cause the potentially problematic non-linearity of pivotality and weights. The *Shapley value* (cf. Shapley 1953) is such an index.[18] It evaluates each representative's chances to be pivotal under the presumption that all political orderings are equiprobable. The latter would be the case in our setting only if $F_1 = \ldots = F_m$. But ideal point orderings are approximately equiprobable inside a suitably small neighborhood of M (cf. Step 3 in Theorem 1's proof). For a given m, one can therefore reduce some approximation inaccuracy associated with following (14) by referring to the Shapley value $\phi_i(q^m; w_1, \ldots, w_m)$ which is induced for constituency C_i by the weighted voting rule $[q^m; w_1, \ldots, w_m]$, instead of only C_i's own weight w_i.

C. Democratic Weights for Independent Constituents

We can specialize the implications of Theorem 1, and the indicated way to improve on equation (14)'s approximation of equal representation for a given m, to the case of mutually independent constituents as follows:

[18] The Shapley value was established as an index of voting power by Shapley and Shubik (1954) on axiomatic grounds. It is often referred to as the *Shapley-Shubik index* in electoral applications. One could interpret Steps 2–3 of our proof as providing additional micro-foundations to it.

Corollary 1 (Square root rule). *If the ideal points of all voters are i.i.d. and delegate i's ideal point equals the median voter's ideal point in constituency C_i for all $i \in \{1, \ldots, m\}$ then choosing*

$$(w_1, \ldots, w_m) \propto \left(\sqrt{n_1}, \ldots, \sqrt{n_m}\right) \tag{15}$$

achieves equal representation asymptotically as $m \to \infty$. For given m, the approximation of perfectly equal representation is typically improved by choosing

$$(w_1, \ldots, w_m) \text{ such that } \phi(q^m; w_1, \ldots, w_m) \propto \left(\sqrt{n_1}, \ldots, \sqrt{n_m}\right). \tag{16}$$

The main part of the corollary, i.e., the reference to the square roots of population sizes, follows because λ_i *is asymptotically* (M, σ_i^2)*-normally distributed*, where $\sigma_i^2 = 1/(n_i[2g(\mathrm{M})]^2)$, if all voter preferences are mutually independent (so $f \equiv g$). This is implied by equations (8)–(9) and the central limit theorem. We therefore have

$$f_i(\mathrm{M}) \approx \frac{1}{\sqrt{2\pi \cdot \frac{1}{n_i[2g(\mathrm{M})]^2}}} = \frac{g(\mathrm{M})}{\sqrt{\pi/2}} \sqrt{n_i} > 0 \tag{17}$$

and can apply this normal approximation to equation (14).[19]

In other words, in the i.i.d. case, the standard deviations of representatives' ideal points $\lambda_1, \ldots, \lambda_m$ are inversely proportional to the square roots of constituency sizes. Since λ_i's (approximately normal) density at the common expected median M is inversely proportional to its standard deviation, the constituency's probability to be the assembly's *unweighted* median is proportional to the square root of its size. That is: the representative of a constituency C_i which is four times larger than constituency C_j has twice the chances to find himself in the middle of his peers. Weights proportional to population sizes would then give C_i more a priori influence than is due. Flat weights would discriminate against C_i because its representative would be pivotal more often than his peers, but not proportionally more often. The balance is struck by relating to the square root of population size.

[19]The error in approximation (17) vanishes quickly enough in order to conclude (15) from (14) also for moderately big population sizes $n_i \ll \infty$. In case of odd n_i, for instance, a well-known approximation of central binomial coefficients implies

$$\begin{aligned} f_i(\mathrm{M}) &= \binom{n_i - 1}{(n_i - 1)/2} \cdot n_i G(\mathrm{M})^{(n_i-1)/2}[1 - G(\mathrm{M})]^{(n_i-1)/2} g(\mathrm{M}) \\ &= \left(1 + O\left(\tfrac{1}{n_i}\right)\right) \frac{4^{(n_i-1)/2}}{\sqrt{\pi(n_i - 1)/2}} \cdot \frac{n_i}{2^{n_i-1}} g(\mathrm{M}) = \frac{g(\mathrm{M})}{\sqrt{\pi/2}} \cdot \sqrt{n_i} + O\left(\tfrac{1}{\sqrt{n_i}}\right). \end{aligned}$$

See, e.g., Arnold et al. (1992, p. 10), for a derivation of density f_i.

D. Discussion

Corollary 1 echoes the finding of Penrose (1946) for the important i.i.d. benchmark case. One is tempted to suspect a deep common reason for this, rather than coincidence. But the respective square root results come about very differently in both models. In the original binary setting, the non-linearity derives from the bottom tier: individual pivot probabilities asymptotically fall in $\sqrt{n_i}$ rather than n_i; square root weights 'correct' for this at the top tier.[20] In the interval case, individual influences are inversely proportional to n_i at the bottom level. So voters from large constituencies indeed suffer the intuitive linear rather than square root 'bottom-tier pivotality disadvantage'. However, the distribution of their delegate's ideal point changes in n_i – in contrast to a constant two-point distribution for all delegates in the binomial case. This change creates a top-tier 'centrality advantage' for delegates from large constituencies in the interval case, which is absent in the binary setting. The top-tier advantage increases in $\sqrt{n_i}$. So square root weights suffice in the interval setting to overcome a citizen's bottom-tier disadvantage.

Analysis of cases in between the binary and interval ones, say, when individual ideal points are independent and identically distributed over $2 < s < \infty$ policy alternatives is very cumbersome. We have checked that voters' bottom-tier disadvantage increases from square root ($s = 2$) to linear ($s = \infty$) at a speed which *differs* from that at which the indicated top-tier centrality advantage changes from non-existent to proportional to $\sqrt{n_i}$. So while the superposition of bottom and top-tier effects happens to imply a square root rule for i.i.d. voter preferences in both the binomial case and the interval case, the same does *not* in general apply to the multinomial case.[21] This raises a first flag concerning any policy recommendations which are justified by Corollary 1 and its binary analogues.

IV. Heterogeneity Within vs. Across Constituencies

A second flag is prompted when we investigate the robustness of the square root rule regarding preference dependence. Adding a non-degenerate constituency-specific shock μ_i to voters' idiosyncratic preference components ϵ^l creates positive correlation of the ideal points $\nu^l = \mu_i + \epsilon^l$ within a constituency. This reflects an often natural *polarization of preferences* along constituency lines, which we can measure by ratio

[20]Note that Penrose's square root rule does not directly relate to weights but top-tier pivot probabilities, which in a binomial voting model equal the *(Penrose-)Banzhaf power index*. See Felsenthal and Machover (1998) or Laruelle and Valenciano (2008b) for good overviews.

[21]There are regular chains $\mathcal{R}^1 \subset \mathcal{R}^2 \subset \mathcal{R}^3 \subset \ldots$ such that square root rules apply to the binary and interval cases while optimal assignments $w_i = c \cdot n_i^{\alpha}$ involve $\alpha > \frac{1}{2} + \delta$ or $\alpha < \frac{1}{2} - \delta$ with $\delta > 0$ for a range of discrete policy spaces in between.

σ_H^2/σ_G^2.[22] It turns out that for $\sigma_H^2/\sigma_G^2 > 0$, a *linear* weight allocation rule quickly performs better than strictly concave mappings.

A. Illustration

This is most easily demonstrated for the case in which the distributions of ϵ^l and μ_i are normal with zero means. Then, if ϵ^l has variance σ_G^2, the median of $\{\epsilon^l\}_{l\in C_i}$ is approximately normal with variance $\pi\sigma_G^2/(2n_i)$. If the constituency-specific preference component μ_i has variance σ_H^2, then constituency C_i's aggregate ideal point λ_i is approximately normal with variance $\pi\sigma_G^2/(2n_i)+\sigma_H^2$ (cf. Arnold et al. 1992, Thm. 8.5.1). Considering the corresponding densities at M $= 0$ for two representatives i and j yields

$$\frac{f_i(0)}{f_j(0)} = \left(\frac{\frac{\pi\sigma_G^2}{2n_i}+\sigma_H^2}{\frac{\pi\sigma_G^2}{2n_j}+\sigma_H^2}\right)^{-\frac{1}{2}}. \tag{18}$$

This quickly approaches 1 if $\sigma_H^2 > 0$ and $n_i, n_j \to \infty$, or if $\sigma_H^2 \to \infty$. Theorem 1 then calls for proportionality of weights to population sizes, not their square roots.

Different population sizes generally give rise to different median distributions. In particular, C_j's distribution F_j is a mean-preserving spread of C_i's F_i if $n_i > n_j$. This is illustrated by Figure 1. It depicts the density functions of ideal points λ_i and λ_j when C_i is four times larger than constituency C_j, so that we seek to achieve $\pi_i(\mathcal{R}^m) = 4\pi_j(\mathcal{R}^m)$ by the choice of suitable voting weights. Panel (a) shows the densities when $\sigma_H^2 = 0$. $f_i(\mathrm{M})/f_j(\mathrm{M}) \approx 2$, and so fair weights satisfy $w_i/w_j \approx 2$. Panel (b) depicts the case when $\mu_i, \mu_j \sim \mathbf{U}[-6\sigma, 6\sigma]$, where $\sigma \equiv \sigma_j$ denotes the standard deviation of the median of idiosyncratic preference components $\{\epsilon^l\}_{l\in C_j}$ in C_j. Then $f_i(\mathrm{M})/f_j(\mathrm{M}) \approx 1$, and $w_i/w_j \approx n_i/n_j = 4$ is implied by Theorem 1 for large m.

Distances between f_i and f_j in panel (b) are globally very small. The densities are almost identical also at the comparatively remote positions which may be locations of the assembly's Condorcet winner for few constituencies, i.e., small m. For the purpose of identifying a representative's odds of being pivotal in $\mathcal{R}^m$, the orderings of the representatives can consequently be treated as having almost equal probabilities, and in this case representative i's Shapley value $\phi_i(q^m; w_1, \ldots, w_m)$ and pivot probability $\pi_i(\mathcal{R}^m)$ are approximately equal. This suggests that ensuring proportionality of the Shapley value to population sizes will solve our problem – independently of the limit considerations in Theorem 1 – for a sufficiently high degree of polarization.

Figure 2 shows the phase transition between a square root rule to a linear rule if we

[22] The basic features of *polarization* according to Esteban and Ray (1994, p. 824) are: (i) a high degree of homogeneity *within* groups, (ii) a high degree of heterogeneity *across* groups, and (iii) a small number of significantly sized groups.

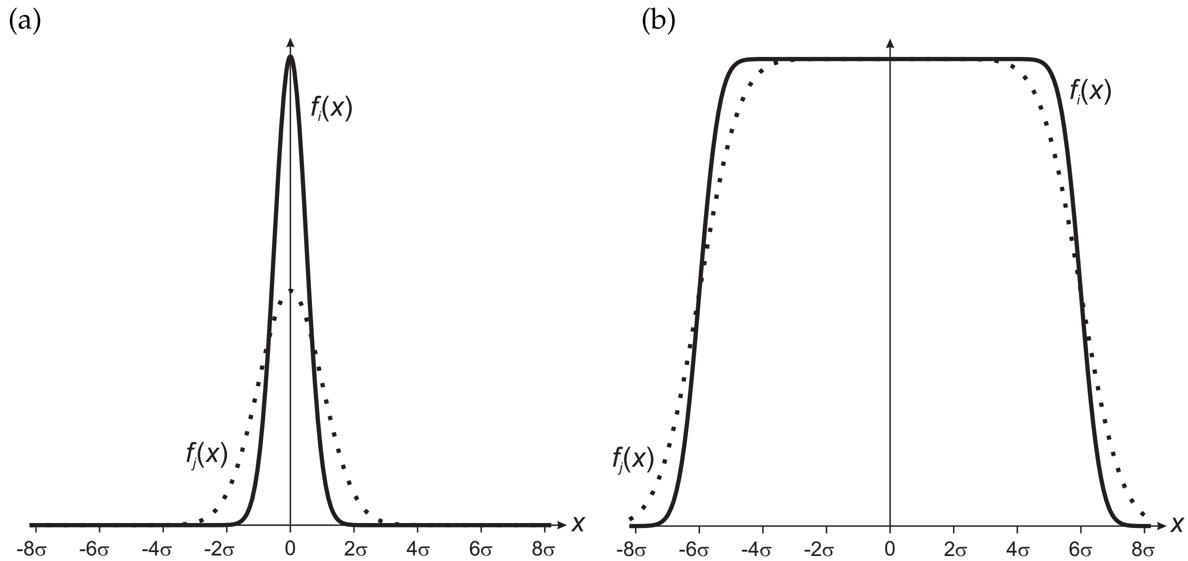


Figure 1: Densities of λ_i and λ_j when $n_i = 4n_j$ and (a) $\mu_i = \mu_j = 0$ or (b) $\mu_i, \mu_j \sim \mathbf{U}[-6\sigma, 6\sigma]$

gradually raise the assumed value of σ_H^2/σ_G^2, and that this transition can be fast. The two panels consider (a) the US population's partition into 50 states and the District of Columbia, and (b) the current European Union with 28 member states (EU28). The dashed lines illustrate the (interpolated) coefficients α^* which are optimal in the sense of minimizing $\|\cdot\|_1$-distance between individual influences and the democratic ideal of $(1/n, \ldots, 1/n) \in \mathbb{R}^n$ as a function of polarization in the class of weighting rules

$$(w_1, \ldots, w_m) \propto ({n_1}^{\alpha}, \ldots, {n_m}^{\alpha}) \tag{19}$$

for $\alpha \in \{0, 0.01, \ldots, 1.99, 2\}$;[23] the solid lines analogously depict the distance-minimizer α^* when we search in the class of Shapley value-based rules

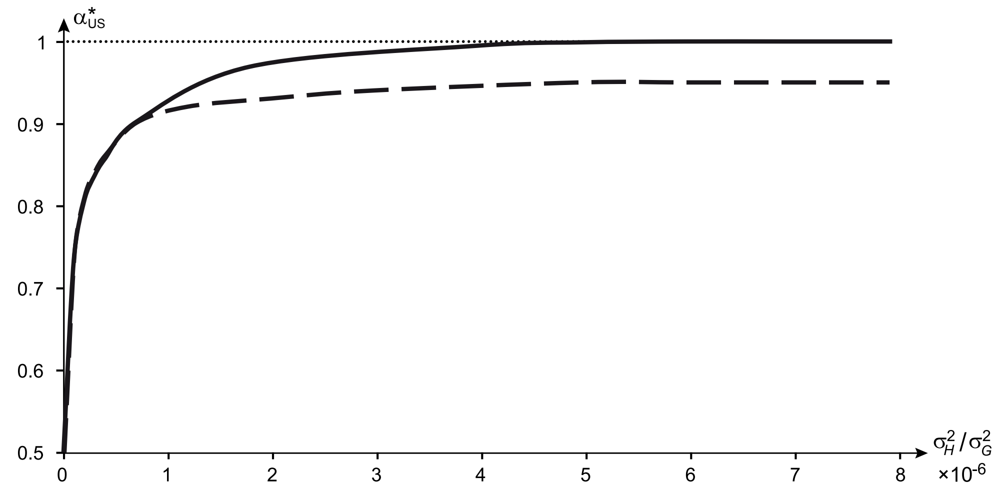
$$(w_1, \ldots, w_m) \text{ such that } \phi(q^m; w_1, \ldots, w_m) \propto ({n_1}^{\alpha}, \ldots, {n_m}^{\alpha})\,. \tag{20}$$

Optimality of the square root rule can be seen to break down quickly. Even small degrees of preference dissimilarity across constituencies render a *linear rule* based on the Shapley value optimal.[24] A qualitative assessment of polarization – are we facing $\sigma_H^2/\sigma_G^2 > 0$ or $\sigma_H^2/\sigma_G^2 \approx 0$? – should hence suffice to produce linear or square root design recommendations in most applications.

[23] We presume $\epsilon^l \sim \mathbf{U}[-0.5, 0.5]$, $\mu_i \sim \mathbf{N}(0, \sigma_H^2)$ with $0 \leq \sigma_H^2 \leq 10^{-6}$ and determine estimates of the pivot probabilities $\pi_i(\mathcal{R}^{28})$ which are induced by a given value of α via Monte Carlo simulation.

[24] We remark that Figure 1(b) illustrated a polarization ratio $\sigma_H^2/\sigma_G^2 = 6\pi/n_j$: $\mathbf{U}[-6\sigma, 6\sigma]$ has variance $\sigma_H^2 = 12\sigma^2$; if $\epsilon^l \sim \mathbf{N}(0, \sigma_G^2)$ then $\sigma_j = \sigma$ corresponds to $\sigma_G^2 = (2n_j/\pi) \cdot \sigma^2$. So the depicted ratio is puny if constituency sizes of US states or EU members are inserted. Also note that α^* fails to converge to 1 in Figure 2 when the simpler weight-based rules in (19) are concerned. This attests to the combinatorial nature of weighted voting, which cannot totally be ignored even for $m = 28$ or $m = 51$.

(a)



(b)

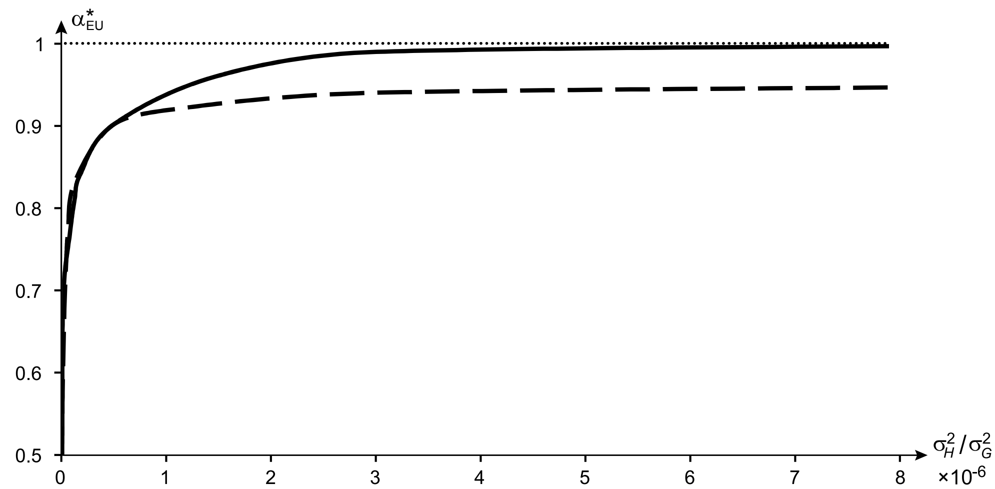


Figure 2: Best coefficient α for direct (dashed line) and Shapley value-based weight allocation rules (solid line) with (a) $n_1, \ldots, n_{51}$ defined by US population data and (b) $n_1, \ldots, n_{28}$ defined by EU28 population data

B. Supermajority Rules

Figure 2(b) considered real EU population data with simple majority rule at the top tier, as in Sections II–III, while decisions to modify the status quo traditionally require a supermajority in the actual EU Council. The decision quota determines the location of the decisive assembly member: it lies in the political center for a 50% threshold but moves towards the legislative status quo for higher quotas (as less and less enthusiastic supporters of change need to be included in order to meet the threshold). This can reverse the locational advantage of representative i over j in the i.i.d. case (see Figure 1(a) for $|x| \geq 0.7\sigma$). The associated square root recommendation breaks down. If, however, median densities agree as in Figure 1(b), then shifts of the status quo-dependent expected location of x^* to the left or right have negligible effect on a constituency's pivot probability. This prompts the following conjecture: not only should a linear Shapley rule apply even for small m – it should hold independently of the applicable majority threshold, provided constituencies are heterogeneous enough. We will prove this below. The finding adds to the practical appeal of proportional weights.

It should be noted, however, that analysis of supermajority rules is based on a weaker notion of decisiveness compared to situations with a threshold of 50%. The reason is that supermajority rules induce entire intervals of undominated polices, instead of a single Condorcet winner. We may nevertheless generalize the quota definition in equation (3) to

$$q^m \equiv q \sum_{j=1}^{m} w_j \;\text{ for }\; q \in [0.5; 1) \tag{21}$$

and consider the representative $P\colon m$ defined by (4) to be at least weakly pivotal. This can be justified by supposing a Pareto inefficient legislative status quo $x^\circ \approx \infty$ and that formation of a winning coalition proceeds as in many motivations of the Shapley value: it starts with the most enthusiastic supporter of change (member $1\colon m$ of the assembly), iteratively including more conservative representatives, and gives all bargaining power to the first – and least enthusiastic – member $P\colon m$ who brings about the required majority.[25]

C. Pivot Probabilities for Polarized Constituencies

We use $\mathcal{R}^{m,q}$ to denote an m-member assembly $\mathcal{R}^m$ which uses the relative decision quota $q \in [0.5; 1)$ and chooses policy $x^* = \lambda_{P:m}$ as defined by equations (4)–(5) and (21). If $q > 0.5$ then the corresponding pivot probabilities $\pi_i(\mathcal{R}^{m,q})$ and $\pi_j(\mathcal{R}^{m,q})$ of representatives i and j in general *fail* to exhibit the limit behavior characterized in Theorem 1; so suggestion (14) and Corollary 1 do not apply.

However, a second asymptotic relationship holds for $q = 0.5$ as well as $q \in (0.5; 1)$. It applies to an arbitrary fixed m and concerns the situation in which independent identically distributed shock variables $\mu_1, \ldots, \mu_m$ are scaled by a factor $t \geq 0$. That is, individual ideal points are given by

$$\nu^l = t \cdot \mu_i + \epsilon^l \tag{22}$$

[25]Status quo x° might also vary randomly on X in this story. Then, if the respective distribution is symmetric around M, $\pi_i(\mathcal{R}^m)$ is i's pivot probability conditional on policy change. Justifications for attributing most or all influence in $\mathcal{R}^m$ to representative $P\colon m$ in the supermajority case date back to Black (1948). The focus on the core's extreme points can be motivated, e.g., by distance-dependent costs of policy reform.

and the corresponding ideal point of representative i from constituency C_i is

$$\lambda_i = t \cdot \mu_i + \tilde{\epsilon}_i \tag{23}$$

with

$$\tilde{\epsilon}_i = \text{median}\,\{\epsilon^l : l \in C_i\}. \tag{24}$$

The i.i.d. benchmark amounts to $t = 0$; a large parameter t corresponds to an electorate which is highly polarized along constituency lines.

If we denote the corresponding pivot probability of representative i by $\pi_i(\mathcal{R}^{m,q,t})$ and abbreviate the Shapley value of the weighted voting game $v = [q^m; w_1, \ldots, w_m]$ as $\phi(v)$, the following holds:

Theorem 2. *Consider an assembly $\mathcal{R}^{m,q}$ with an arbitrary number m of constituencies and relative decision quota $q \in [0.5; 1)$. Let the ideal point of each representative $i \in \{1, \ldots, m\}$ be $\lambda_i = t \cdot \mu_i + \tilde{\epsilon}_i$, and suppose $\mu_1, \ldots, \mu_m$ and $\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_m$ are mutually independent, $\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_m$ have finite second moments, and $\mu_1, \ldots, \mu_m$ have identical bounded densities. Then*

$$\lim_{t \to \infty} \frac{\pi_i(\mathcal{R}^{m,q,t})}{\pi_j(\mathcal{R}^{m,q,t})} = \frac{\phi_i(v)}{\phi_j(v)} \tag{25}$$

supposing that $\phi_j(v) > 0$.

The proof is provided in Appendix 2. This result does not presume equation (24) to hold. In contrast to Theorem 1, it places no restrictions on the densities of $\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_m$ nor on the voting weights $w_1, \ldots, w_m$ in assembly $\mathcal{R}^{m,q}$.

D. Democratic Weights for Affiliated Constituents

Applied to our model of democratic representation, variables $\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_m$ in Theorem 2 correspond to the medians of $n_1, \ldots, n_m$ draws of i.i.d. idiosyncratic preference components ϵ^l. As we already exploited in equation (18), the variance of $\tilde{\epsilon}_i$ is approximately equal to $\pi\sigma_G^2/(2n_i)$. This means that the effect of idiosyncratic noise on C_i's median opinion can essentially be ignored for constituency sizes in the thousands or millions: λ_i's variance of approximately $\pi\sigma_G^2/(2n_i)+\sigma_H^2$ is dominated by μ_i's variance σ_H^2 unless the latter is smaller than σ_G^2 by several orders of magnitude. This implies:

Corollary 2 (Linear Shapley rule). *If individual ideal points are the sum of i.i.d. idiosyncratic components and i.i.d. constituency components with similar orders of magnitude then*

$$(w_1, \ldots, w_m) \text{ such that } \phi(q^m; w_1, \ldots, w_m) \propto (n_1, \ldots, n_m) \tag{26}$$

achieves approximately equal representation for any given relative decision quota $q \in [0.5; 1)$ if constituency populations are large.

We note that the Shapley value $\phi(v)$ of voting game $v = [q^m; w_1, \ldots, w_m]$ is often close to the relative weight vector $(w_1, \ldots, w_m)/\sum_i w_i$ (see, e.g., Jelnov and Tauman 2014). Using population shares as voting weights is therefore a good practical default for implementing (26). We only caution that it *can* involve considerable avoidable errors when m is small, the distribution of constituency sizes is very skewed, or q is close to 1. These cases are prone to pronounced non-linearity between voting weight and power.

For instance, there exist only 9 structurally different weighted voting games (up to isomorphisms) in case of $m = 4$ and simple majority quota $q = 0.5$. Numbers in the corresponding Shapley values $\phi(v)$ must be multiples of $1/4! = 4.1\bar{6}\%$.[26] Exact proportionality to population shares of, say, $\bar{\mathbf{n}} = (42\%, 25\%, 24\%, 9\%)$ can, therefore, not be achieved – one needs to live with pivot probabilities which approximate $\bar{\mathbf{n}}$. Default weights $(w_1, \ldots, w_4) = \bar{\mathbf{n}}$ in this example induce pivot probabilities of $\pi(\mathcal{R}^4) \approx \phi(v) = (50\%, 16.\bar{6}\%, 16.\bar{6}\%, 16.\bar{6}\%)$. This is arguably not a very satisfactory approximation. In particular, it is more distant from $\bar{\mathbf{n}}$ than $\pi(\mathcal{R}^4) \approx \phi(v') = (41.\bar{6}\%, 25\%, 25\%, 8.\bar{3}\%)$, which would be induced by $(w'_1, \ldots, w'_4) = (40\%, 25\%, 25\%, 10\%)$. In the light of this, one ideally tries to solve the problem

$$\min_{\mathbf{w}} \|\phi(q^m; \mathbf{w}) - \bar{\mathbf{n}}\| \tag{27}$$

for a suitable norm $\|\cdot\|$, at least when $\|\phi(q^m; \bar{\mathbf{n}}) - \bar{\mathbf{n}}\|$ looks big. This is the so-called *inverse problem* for the Shapely value: to find voting weights which induce a desired Shapley vector, or come as close as possible. It is non-trivial, but has been studied.[27]

E. Discussion

Whether Corollary 2 for the case of affiliated constituents or Corollary 1 for the i.i.d. case provides better guidance for designing a fair two-tier voting system is obviously a contingent matter. Some preference homogeneity within, and dissimilarity across, constituencies seems very plausible. It can arise as the result of a sorting process ('voting with one's feet') à la Tiebout (1956), be due to cultural uniformity fostered by proximity and local interaction (see Alesina and Spolaore 2003), or have other

[26]Recall that

$$\phi_i(v) \equiv \sum_{S \subseteq \{1,\ldots,m\} \setminus \{i\}} \frac{|S|! \cdot (m - |S| - 1)!}{m!} [v(S \cup \{i\}) - v(S)]$$

where worth $v(S)$ of coalition S is 1 if $\sum_{i \in S} w_i > q^m$, and 0 otherwise (Shapley 1953).

[27]Complete enumeration of voting games is the main option for $m < 9$. Kurz (2012) shows how *integer linear programming* techniques can be brought to bear instead, but exact solutions are still computationally demanding for $m > 10$. Good heuristic solutions exist, especially if the relative quota q is a variable rather than given. See, e.g., Kurz and Napel (2014). Solutions to problem (27) may still involve non-negligible distances: for instance, the Shapley vector with minimal $\|\cdot\|_1$-distance to $\bar{\mathbf{n}} = (49\%, 33\%, 9\%, 9\%)$ is $(41.\bar{6}\%, 25\%, 25\%, 8.\bar{3}\%)$.

reasons. If constituencies correspond to entire nations, as in case of the EU Council or ECB Governing Council, members of a given constituency typically share more historical experience, traditions, language, communication etc. within constituencies than across. (This seems the key practical reason for why the issue of asymmetric constituency sizes cannot trivially be resolved by redistricting.) This speaks clearly in favor of a linear rule. Still, the collective decisions that are taken by the top-tier assembly might be primarily about issues where opinions range over identical liberal–conservative or state–market spectrums in all constituencies. Moreover, there might be normative reasons outside the scope of our analysis for setting $\sigma_H^2 = 0$ when one designs a constitution. We therefore avoid specific recommendations here for, say, new voting rules in the EU Council.[28] However, we warn that the i.i.d. presumption is considerably more knife-edged and hence requires special motivation.

V. Concluding Remarks

We have extended the classical binary analysis of collective choice in two-tier systems to a median voter world with a continuum of alternatives. Our results broaden the basis for a priori assessments of voting weight arrangements, which differ widely in practice. Arguably, the informal balance of power between constituencies at the time of setting up a system of divisional preference aggregation matters more for the selected voting rule than normative arguments like ours. Still, such arguments have occasionally been taken up by practitioners (see, e.g., the Swedish diplomat Moberg 2010 on voting rules in the EU). In any case they clarify the premises behind competing claims that this or that system – plain proportionality, various forms of 'degressive proportionality', complete disregard for constituency size differences – is fairer than another.

Our analysis has taken up the basic democratic principle of 'one person, one vote' and operationalized it as requiring proportionality between a constituency's size and the respective probability of getting its way. It turns out that this objective calls for approximate proportionality of voting weights to the square root of population sizes if individual voters' single-peaked preferences vary independently between and within constituencies. This finding is, however, not very robust. The more intuitive linear recommendation obtains if intra-constituency preference similarities are taken into account. Then, by default, weights ought to be proportional to population sizes. Adaptations which target a proportional Shapley value are even more equal.

[28] An interesting option, inspired by the call for "flexible" democratic mechanisms in other contexts (see Gersbach 2005, 2009), would be to specify different voting rules for different policy domains of the EU. In some areas, such as competition policy, small or unstable between-constituency differences may call for square root weights; while fair decision making in other policy domains, such as agriculture or fisheries – with heterogeneous shares of farmable land and sea access – could involve linear weights.

Although the equality of citizens' a priori influence on collective choices is a desirable ideal, it is of course not the only relevant benchmark. Pursuit of a welfarist design objective (find weights such that expected total utility of voters is maximized) and the majoritarian goal of minimizing the expected 'democracy gap' between the two-tier policy outcome and the outcome preferred by the population median come to mind.

Simulations suggest that the policy implications of all these objectives are actually quite similar (see Maaser and Napel 2012, 2014). The different ideals are also linked theoretically: the hypothetical situation in which outcomes of a representative system perfectly imitate the predicted outcomes $x^{**} \equiv \text{median}\{v^l : l = 1, \ldots, n\}$ of direct democracy would necessarily involve proportionality of a constituency's pivot probability and its size for independent and identically distributed preferences. Welfare-maximizing voting weights would, similarly, try to bring the two-tier outcome x^* in congruence with the population's sample median if voters' utilities decrease linearly in distance to their respective ideals. So there are reasons to conjecture that our conclusions extend to interesting alternative desiderata. (If utility falls quadratically, and so total welfare is maximized by the sample mean, symmetry of the ideal point distributions would still make x^{**} an attractive target.)

Unfortunately, coincidence of the pivot probabilities in ideal situations with the proportional pivot probabilities which Corollaries 1 and 2 seek to implement just provide a suggestive heuristic. Identity of x^* and x^{**} is, in general, unachievable. Pivot probabilities in the best feasible approximations may differ; so separate formal arguments are needed for each normative goal. Our Theorems 1 and 2 may help deriving them, as may the von Mises calculus which we invoked in order to operationalize individual influence (cf. fn. 12). The associated functional analytic methods look difficult, at least to us, and weighted order statistics from non-identical distributions are generally unwieldy. We still hope that it will be possible to move beyond the prevailing binary focus on indirect collective choice, also for design objectives other than the one addressed here. It is an open challenge to obtain analogous results on the utilitarian and the majoritarian weights of nations.

Mathematical Appendix

→ Possibly to be published Online ←

1. Proof of Theorem 1

1.A Overview

The major observation in the proof of Theorem 1 is that, as m grows large, the pivotal member of $\mathcal{R}^m$ is most likely found very close to the median M of distributions $F_1, \ldots, F_m$. Namely, the probability for the realized weighted median in $\mathcal{R}^m$ to fall outside a neighborhood of M turns out to approach zero at an exponential speed. One can therefore restrict attention to a neighborhood N_ε of M. In contrast to the more and more deterministic location of $\mathcal{R}^m$'s pivotal member,[29] the pivotal representative's identity remains a complicated, weight-dependent random variable even as $m \to \infty$. However, orderings of those representatives with ideal points inside N_ε become in good approximation conditionally equiprobable. Delegate i's conditional pivot probability, therefore, corresponds to i's Shapley value in 'subgames' which involve only the representatives j with realizations $\lambda_j \in N_\varepsilon$. It is then possible to apply the limit result proven by Neyman (1982) for the Shapley value and to exploit that the probability of condition $\{\lambda_i \in N_\varepsilon\}$ being true becomes proportional to λ_i's density at M when $\varepsilon \downarrow 0$.

The precise argument is structured into five steps. In *Step 1*, we define a particular neighborhood I_m of the expected location of the weighted median of $\lambda_1, \ldots, \lambda_m$. This *essential interval* I_m shrinks to $\{\text{M}\}$ as $m \to \infty$. It is constructed such that the probabilities p_θ, $\overset{\sqsubset}{p}_\theta$, and $\overset{\sqsupset}{p}_\theta$ of a type-θ representative's ideal point falling inside I_m, inside I_m's left half, or inside I_m's right half, respectively, can suitably be bounded. Moreover, we decompose the deterministic total number $m_\theta = \beta_\theta(m) \cdot m$ of type-θ representatives in assembly $\mathcal{R}^m$ into the random numbers ${}^{\triangleleft}k_\theta$, k_θ, and $k^{\triangleright}_\theta$ of delegates with ideal points to I_m's left, inside I_m, and to I_m's right. Knowing the respective vector $\mathbf{k} = ({}^{\triangleleft}k_1, k_1, k^{\triangleright}_1, \ldots, {}^{\triangleleft}k_r, k_r, k^{\triangleright}_r)$ is sufficient to determine whether the Condorcet winner is located inside I_m or not.

In *Step 2*, it is established that the weighted median of $\lambda_1, \ldots, \lambda_m$ is located inside the essential interval I_m with a probability that exponentially approaches 1 as $m \to \infty$. As a corollary, the probability $\pi^\theta(\mathcal{R}^m)$ of the Condorcet winner having type θ converges to the corresponding *conditional* probability $\pi^\theta(\mathcal{R}^m|\mathcal{K})$ of a type-θ representative being

[29]If one took the assumptions of known preference distributions and an unbounded number m of constituencies literally, someone might suggest to dispense with voting and simply implement M. The limit consideration is, of course, only an analytical device. Numerical investigations, e.g., by Maaser and Napel (2007) confirm that the asymptotic findings are already a good guide for $10 < m < 50$.

pivotal where event $\mathcal{K}$ comprises all realizations of $\mathbf{k}$ such that $\mathcal{R}^m$'s weighted median lies inside I_m.

In *Step 3*, we show that the random orderings of the $k = \sum_{\theta \in \{1,\ldots,r\}} k_\theta$ representatives with ideal point realizations $\lambda_i \in I_m$ asymptotically become equiprobable as $m \to \infty$. It follows that, with a vanishing error, the respective conditional pivot probability $\pi^\theta(\mathcal{R}^m|\mathcal{K})$ equals the expected aggregate Shapley value of type-θ representatives in I_m.

In *Step 4*, the strong convergence result by Neyman (1982) is applied to our setting. Neyman's result implies that the aggregate Shapley value of type-θ representatives with ideal points in I_m converges to their respective aggregate voting weight in each considered weighted voting 'subgame' among the representatives with ideal points $\lambda_i \in I_m$.

Having established that $\pi^\theta(\mathcal{R}^m)$ is asymptotically proportional to the aggregate voting weight of all type-θ representatives with ideal points inside I_m, aggregate probabilities are broken down to individual representatives in the final *Step 5*.

1.B Proof

Step 1: Essential interval I_m and vector k

We begin by identifying a neighborhood of M and a sufficiently great number m such that both the densities f_θ and the numbers of type-θ representatives in $\mathcal{R}^m$ can suitably be bounded. This leads to the definition of intervals I_m around M which later steps will focus on. Bounds for the probabilities of a type-θ representative's ideal point falling inside I_m, and more specifically into I_m's left or right halves, are provided in Lemma 1. The final part of *Step 1* introduces the vector $\mathbf{k}$ as a type-specific summary of how many ideal points are located to the left of I_m, inside I_m, and to its right.

First note that

$$0 < \underline{u} \equiv \min_{\theta' \in \{1,\ldots,r\}} f_{\theta'}(\mathrm{M}) \leq f_\theta(\mathrm{M}) \leq \overline{u} \equiv \max_{\theta' \in \{1,\ldots,r\}} f_{\theta'}(\mathrm{M}) \tag{28}$$

for every $\theta \in \{1, \ldots, r\}$. Using the continuity of f_θ in a neighborhood $(\mathrm{M} - \varepsilon_1, \mathrm{M} + \varepsilon_1)$ of M, which is implied by $|f_\theta(x) - f_\theta(\mathrm{M})| \leq c(x - \mathrm{M})^2$, we can choose $0 < \varepsilon_2 \leq \varepsilon_1$ such that

$$\frac{5}{6} f_\theta(\mathrm{M}) \leq f_\theta(x) \leq \frac{7}{6} f_\theta(\mathrm{M}) \tag{29}$$

for all $x \in [\mathrm{M} - \varepsilon_2, \mathrm{M} + \varepsilon_2]$ and any specific $\theta \in \{1, \ldots, r\}$. Inequality (28) can be used in order to obtain bounds

$$\frac{1}{2} \underline{u} \leq f_\theta(x) \leq 2\overline{u} \tag{30}$$

for all $x \in [\mathrm{M} - \varepsilon_2, \mathrm{M} + \varepsilon_2]$ and all $\theta \in \{1, \ldots, r\}$ which do not depend on θ. The assumed regularity of $\mathcal{R}^1 \subset \mathcal{R}^2 \subset \mathcal{R}^3 \subset \ldots$ entails the existence of some $m^0 \in \mathbb{N}$ such that $\beta_\theta(m) \geq \beta > 0$ for all $m \geq m^0$. So we can also choose $0 < \varepsilon_3 \leq \varepsilon_2$ such that

$$\beta_\theta(m) \geq \beta > 0 \tag{31}$$

for all $m \geq \frac{1}{\varepsilon_3{}^{8/3}}$ and all $\theta \in \{1, \ldots, r\}$. And we can determine $0 < \varepsilon_4 \leq \varepsilon_3$ such that

$$24 < \underline{u}\beta \cdot (m\beta)^{\frac{1}{40}} \leq \underline{u}\beta m_\theta^{\frac{1}{40}} \tag{32}$$

for all $m \geq \frac{1}{\varepsilon_4{}^{8/3}}$, where $m_\theta \equiv \beta_\theta(m) \cdot m$.

Then define

$$\varepsilon(m) \equiv m^{-\frac{3}{8}} \tag{33}$$

and note that $\varepsilon(m) \leq \varepsilon_4$ iff $m \geq m^1 \equiv \frac{1}{\varepsilon_4{}^{8/3}} \geq m^0$. So, whenever we consider a sufficiently large number of representatives (specifically, $m \geq m^1$), inequalities (29)–(32) are satisfied. We refer to

$$I_m \equiv [\mathrm{M} - \varepsilon(m), \mathrm{M} + \varepsilon(m)] \tag{34}$$

as the *essential interval*. The probability of an ideal point of type θ to fall inside I_m is

$$p_\theta \equiv \int\limits_{\mathrm{M}-\varepsilon(m)}^{\mathrm{M}+\varepsilon(m)} f_\theta(x)dx. \tag{35}$$

For realizations in the left and right halves of I_m we respectively obtain the probabilities

$$\overset{\sqsubset}{p}_\theta \equiv \int\limits_{\mathrm{M}-\varepsilon(m)}^{\mathrm{M}} f_\theta(x)dx \quad \text{and} \quad \overset{\sqsupset}{p}_\theta \equiv \int\limits_{\mathrm{M}}^{\mathrm{M}+\varepsilon(m)} f_\theta(x)dx, \tag{36}$$

with $\overset{\sqsubset}{p}_\theta + \overset{\sqsupset}{p}_\theta = p_\theta$.

Lemma 1. *For $m \geq m^1$ we have*

$$\frac{5}{3} f_\theta(\mathrm{M})\varepsilon(m) \leq \quad p_\theta \quad \leq \frac{7}{3} f_\theta(\mathrm{M})\varepsilon(m), \tag{37}$$

$$\frac{5}{6} f_\theta(\mathrm{M})\varepsilon(m) \leq \quad \overset{\sqsubset}{p}_\theta, \overset{\sqsupset}{p}_\theta \quad \leq \frac{7}{6} f_\theta(\mathrm{M})\varepsilon(m), \tag{38}$$

$$\underline{u}\beta m_\theta^{-\frac{3}{8}} \leq \quad p_\theta \quad \leq 4\overline{u} m_\theta^{-\frac{3}{8}}, \ \textit{and} \tag{39}$$

$$\frac{1}{2}\underline{u}\beta m_\theta^{-\frac{3}{8}} \leq \quad \overset{\sqsubset}{p}_\theta, \overset{\sqsupset}{p}_\theta \quad \leq 2\overline{u} m_\theta^{-\frac{3}{8}}. \tag{40}$$

Proof. The inequalities can be concluded from (29)–(31), $m_\theta = \beta_\theta m$, and $\beta < 1$. $\square$

Now for any realization $\boldsymbol{\lambda}$ of the ideal points in assembly $\mathcal{R}^m$, let

$$k_\theta \equiv \#\{j\colon \tau(j) = \theta \text{ and } \lambda_j \in [\mathrm{M} - \varepsilon(m), \mathrm{M} + \varepsilon(m)]\} \tag{41}$$

denote the number of type-θ representatives with a policy position in the essential interval, i.e., no more than $\varepsilon(m)$ away from the expected sample median M. Analogously, let

$$^{\triangleleft}\!k_\theta \equiv \#\{j\colon \tau(j) = \theta \text{ and } \lambda_j \in (-\infty, \mathrm{M} - \varepsilon(m))\} \tag{42}$$

and

$$k^{\triangleright}_\theta \equiv \#\{j\colon \tau(j) = \theta \text{ and } \lambda_j \in (\mathrm{M} + \varepsilon(m), \infty)\} \tag{43}$$

denote the random number of type-θ representatives to the left and to the right of I_m.

One can conceive of $\boldsymbol{\lambda}$-realizations as the results of two-part random experiments: in the first part, it is determined for each λ_j whether it is located to the right of I_m, to its left, or inside I_m, e.g., by drawing a vector $\boldsymbol{l} = (l_1, \ldots, l_m)$ of independent random variables where $l_i = 1$ (-1) indicates a realization of λ_i to the right (left) of I_m and $l_i = 0$ indicates $\lambda_i \in I_m$ (with probabilities $\frac{1}{2} - \overset{\sqsubset}{p}_\theta$, $\frac{1}{2} - \overset{\sqsupset}{p}_\theta$, and p_θ, respectively). This already fixes $^{\triangleleft}\!k_\theta$, k_θ, and $k^{\triangleright}_\theta$ for each $\theta \in \{1, \ldots, r\}$ and is summarized by the vector

$$\mathbf{k} = (^{\triangleleft}\!k_1, k_1, k^{\triangleright}_1, \ldots, {}^{\triangleleft}\!k_r, k_r, k^{\triangleright}_r). \tag{44}$$

In the second part, the exact ideal point locations are drawn. It will turn out that those outside I_m can be ignored with vanishing error; and the k_θ type-θ ideal points inside have conditional densities $\hat{f}_\theta$ with

$$\hat{f}_\theta(x) \equiv \frac{f_\theta(x)}{p_\theta} \quad \text{for } x \in I_m. \tag{45}$$

Step 2: Type θ's aggregate pivot probability $\pi^\theta(\mathcal{R}^m)$ converges to the conditional probability $\pi^\theta(\mathcal{R}^m|\mathcal{K})$ of type θ being pivotal in I_m

We next appeal to *Hoeffding's inequality*[30] in order to obtain bounds on the probability that the shares of representatives $^{\triangleleft}\!k_\theta/m_\theta$, k_θ/m_θ, and $k^{\triangleright}_\theta/m_\theta$ with ideal points to the left, inside, or right of I_m deviate by more than a specified distance from their expectations. These bounds will imply that one can condition on the pivotal ideal point lying inside I_m in later steps of the proof with an exponentially decreasing error.

[30] See Hoeffding (1963, Theorem 2).

Hoeffding's inequality concerns the average $\overline{X} \equiv \frac{1}{n} \cdot \sum_{i=1}^{n} X_i$ of n independent bounded random variables $X_i \in [a_i, b_i]$ and guarantees

$$\Pr\left\{\left|\overline{X} - \mathbf{E}[\overline{X}]\right| > t\right\} \leq 2 \exp\left(\frac{-2t^2 n^2}{\sum_{i=1}^{n} (b_i - a_i)^2}\right). \tag{46}$$

Our specific construction will involve only random variables $X_i \in [0, 1]$, so that

$$\Pr\left\{\left|\overline{X} - \mathbf{E}[\overline{X}]\right| > t\right\} \leq 2 \exp\left(-2t^2 n\right). \tag{47}$$

We will put $n = m_\theta$ for a fixed $\theta \in \{1, \ldots, r\}$, so that $n \to \infty$ as $m \to \infty$, and choose $t = n^{-\frac{2}{5}}$, which implies $t(n) \ll \varepsilon(m)$. For this choice

$$\Pr\left\{\left|\overline{X} - \mathbf{E}[\overline{X}]\right| > n^{-\frac{2}{5}}\right\} \leq 2 \exp\left(-2n^{\frac{1}{5}}\right), \tag{48}$$

i.e., the probability of "extreme realizations" exponentially goes to zero as $m \to \infty$ (and hence $n = m_\theta \to \infty$).

Lemma 2. *For each $\theta \in \{1, \ldots, r\}$ we have:*

$$\begin{array}{llcl}
\text{(I)} & \Pr\left\{\dfrac{{}^{\triangleleft}k_\theta}{m_\theta} \in \left[\dfrac{1}{2} - \overset{\sqsubset}{p}_\theta - m_\theta^{-\frac{2}{5}}, \dfrac{1}{2} - \overset{\sqsubset}{p}_\theta + m_\theta^{-\frac{2}{5}}\right]\right\} & \geq & 1 - 2\exp\left(-2m_\theta^{\frac{1}{5}}\right) \\
\text{(II)} & \Pr\left\{\dfrac{k_\theta}{m_\theta} \in \left[p_\theta - m_\theta^{-\frac{2}{5}}, p_\theta + m_\theta^{-\frac{2}{5}}\right]\right\} & \geq & 1 - 2\exp\left(-2m_\theta^{\frac{1}{5}}\right) \\
\text{(III)} & \Pr\left\{\dfrac{k^{\triangleright}_\theta}{m_\theta} \in \left[\dfrac{1}{2} - \overset{\sqsupset}{p}_\theta - m_\theta^{-\frac{2}{5}}, \dfrac{1}{2} - \overset{\sqsupset}{p}_\theta + m_\theta^{-\frac{2}{5}}\right]\right\} & \geq & 1 - 2\exp\left(-2m_\theta^{\frac{1}{5}}\right).
\end{array}$$

Proof. Let $\theta \in \{1, \ldots, r\}$ be arbitrary but fixed. For statement (I) we consider the $n = m_\theta$ indices $j_1 \ldots, j_{m_\theta} \in \{1, \ldots, m\}$ of type θ and denote by X_i the random variable which is 1 if the realization λ_{j_i} lies inside the interval $(-\infty, \mathrm{M} - \varepsilon(m))$ and zero otherwise. In the notation of Hoeffding's inequality we have $\overline{X} = {}^{\triangleleft}k_\theta / m_\theta$. Since the probability that λ_{j_i} lies in the left half of I_m is given by $\overset{\sqsubset}{p}_\theta$ and $\int_{-\infty}^{\mathrm{M}} f_\theta(x)dx = \int_{\mathrm{M}}^{\infty} f_\theta(x)dx = \frac{1}{2}$, the probability that λ_{j_i} lies in the interval $(-\infty, \mathrm{M} - \varepsilon(m))$ is given by $\frac{1}{2} - \overset{\sqsubset}{p}_\theta$. Thus we have $\mathbf{E}[\overline{X}] = \frac{1}{2} - \overset{\sqsubset}{p}_\theta$ and (48) implies (I). The statements (II) and (III) follow along the same lines (namely, by letting X_i be the characteristic function of intervals $[\mathrm{M} - \varepsilon(m), \mathrm{M} + \varepsilon(m)]$ and $(\mathrm{M} + \varepsilon(m), \infty)$, respectively). Note that ${m_\theta}^{-2/5} \ll \varepsilon(m) = m^{-3/8}$ for large m. □

We can use the bounds on p_θ in (39) and that $\beta m \leq m_\theta \leq m$ for $m \geq m^1 \geq m^0$ in order to conclude from (II) that for any given $\theta \in \{1, \ldots, r\}$

$$\underline{u}\beta^2 \varepsilon(m) \cdot m - m^{\frac{3}{5}} \le k_\theta \le 4\overline{u}\varepsilon(m) \cdot m + m^{\frac{3}{5}} \tag{49}$$

with a probability of at least $1 - 2 \cdot \exp\left(-2m_\theta{}^{\frac{1}{5}}\right)$. A further implication of observations (I)–(III) is:

Lemma 3. *For $m \ge m^1$ the inequalities*

$$\begin{aligned}
{}^{\triangleleft}k_\theta &< \frac{1}{2}m_\theta && (50)\\
k^{\triangleright}_\theta &< \frac{1}{2}m_\theta && (51)\\
{}^{\triangleleft}k_\theta + \frac{2}{3}k_\theta &> \frac{1}{2}m_\theta && (52)\\
k^{\triangleright}_\theta + \frac{2}{3}k_\theta &> \frac{1}{2}m_\theta && (53)
\end{aligned}$$

are simultaneously satisfied for all $\theta \in \{1, \ldots, r\}$ *with a probability of at least* $1 - 6r \cdot \exp\left(-2(\beta m)^{\frac{1}{5}}\right)$.

Proof. The events considered in statements (I), (II), and (III) of Lemma 2 are realized for *all* $\theta \in \{1, \ldots, r\}$ with a joint probability of at least

$$\left(1 - 2\exp\left(-2(\beta m)^{\frac{1}{5}}\right)\right)^{3r} \ge 1 - 6r\exp\left(-2(\beta m)^{\frac{1}{5}}\right), \tag{54}$$

since $m_\theta \ge \beta m$ for $m \ge m^0$ and $(1-x)^k \ge (1-kx)$ is valid for all $x \in [0,1]$ and $k \in \mathbb{N}$. If $m \ge m^1$, we then have

$${}^{\triangleleft}k_\theta \le \left(\frac{1}{2} - \overset{\sqsubset}{p}_\theta\right) \cdot m_\theta + m_\theta{}^{\frac{3}{5}} \le \frac{m_\theta}{2} - \frac{\underline{u}\beta m_\theta{}^{\frac{5}{8}}}{2} + m_\theta{}^{\frac{3}{5}} = \frac{m_\theta}{2} - m_\theta{}^{\frac{3}{5}} \underbrace{\left(\frac{\underline{u}\beta m_\theta^{\frac{1}{40}}}{2} - 1\right)}_{>0} < \frac{1}{2}m_\theta \tag{55}$$

for any $\theta \in \{1, \ldots, r\}$. The first inequality follows directly from (I), the second inequality uses (40), and the final inequality follows from (32). Analogous inequalities pertain to $k^{\triangleright}_\theta$.

Moreover, we can conclude

$$\begin{aligned}
\overset{\triangleleft}{k}_\theta + \frac{2}{3}k_\theta &\geq \left(\frac{1}{2} - \overset{\sqsubset}{p}_\theta\right) \cdot m_\theta - m_\theta{}^{\frac{3}{5}} + \frac{2p_\theta}{3} m_\theta - \frac{2}{3} m_\theta{}^{\frac{3}{5}} && (56)\\
&= \frac{m_\theta}{2} - \frac{5}{3} m_\theta{}^{\frac{3}{5}} + \left(\frac{2p_\theta}{3} - \overset{\sqsubset}{p}_\theta\right) m_\theta && (57)\\
&= \frac{m_\theta}{2} + \frac{5}{3} m_\theta{}^{\frac{3}{5}} \left(\frac{2\overset{\sqsupset}{p}_\theta}{5} m_\theta{}^{\frac{2}{5}} - \frac{\overset{\sqsubset}{p}_\theta}{5} m_\theta{}^{\frac{2}{5}} - 1\right) && (58)\\
&\geq \frac{m_\theta}{2} + \frac{5}{3} m_\theta{}^{\frac{3}{5}} \left(\frac{1}{10} \cdot \frac{3}{7} p_\theta \cdot m_\theta{}^{\frac{2}{5}} - 1\right) && (59)\\
&\geq \frac{m_\theta}{2} + \frac{5}{3} m_\theta{}^{\frac{3}{5}} \underbrace{\left(\frac{\underline{u}\beta m_\theta{}^{\frac{1}{40}}}{24} - 1\right)}_{>0} > \frac{1}{2} m_\theta. && (60)
\end{aligned}$$

The first inequality uses (I) and (II); the second one employs (37) and (38); the third applies (39); and the final one invokes (32). Analogous inequalities pertain to $\overset{\triangleright}{k}_\theta + \frac{2}{3}k_\theta$. □

Lemma 3 implies that the respective unweighted sample median among representatives of type θ is located within I_m for all $\theta \in \{1, \ldots, r\}$ with a probability that quickly approaches 1. The same must *a fortiori* be true for the pivotal assembly member, i.e., the weighted median among all representatives.

We collect in the set $\mathcal{K}$ all $\mathbf{k} = (\overset{\triangleleft}{k}_1, k_1, \overset{\triangleright}{k}_1, \ldots, \overset{\triangleleft}{k}_r, k_r, \overset{\triangleright}{k}_r)$ such that the events considered by Lemma 2, (I)–(III), are realized for *all* $\theta \in \{1, \ldots, r\}$. The inequalities in Lemma 3 then hold for any $\mathbf{k} \in \mathcal{K}$. We can decompose the probability $\pi^\theta(\mathcal{R}^m)$ of some type-θ representative being pivotal into conditional probabilities $\pi^\theta(\mathcal{R}^m|\mathcal{K})$ and $\pi^\theta(\mathcal{R}^m|\neg\mathcal{K})$ which respectively concern only $\boldsymbol{\lambda}$-realizations where $\mathbf{k} \in \mathcal{K}$ and $\mathbf{k} \notin \mathcal{K}$. Then Lemma 3 implies

$$\begin{aligned}
\pi^\theta(\mathcal{R}^m) &= \Pr\{\mathcal{K}\} \cdot \pi^\theta(\mathcal{R}^m|\mathcal{K}) + \Pr\{\neg\mathcal{K}\} \cdot \pi^\theta(\mathcal{R}^m|\neg\mathcal{K}) \\
&= \pi^\theta(\mathcal{R}^m|\mathcal{K}) + O(exp(-2m^{\frac{1}{5}})). && (61)
\end{aligned}$$

Step 3: $\pi^\theta(\mathcal{R}^m|\mathcal{K})$ converges to the expectation of type θ's Shapley value inside I_m

Now condition on some $\mathbf{k} \in \mathcal{K}$ such that exactly $\sum_\theta k_\theta = k$ ideal points fall inside the essential interval, where k is asymptotically proportional to $\varepsilon(m) \cdot m = m^{\frac{5}{8}}$ by (49). Label them $1, \ldots, k$ for ease of notation and let $\varrho \in \mathcal{S}_k$ denote an arbitrary element of the space $\mathcal{S}_k$ of permutations which bijectively map $(1, \ldots, k)$ to some $(j_1, \ldots, j_k)$. The conditional probability for the event that the k ideal points located in I_m are ordered exactly as they are in ϱ by the second step of the experiment is

$$p(\varrho|\mathbf{k}) \equiv \int_{-\varepsilon(m)}^{\varepsilon(m)} \int_{x_{j_1}}^{\varepsilon(m)} \dots \int_{x_{j_{k-1}}}^{\varepsilon(m)} \hat{f}_{j_1}(x_{j_1}) \dots \hat{f}_{j_k}(x_{j_k})\, dx_{j_k} \dots dx_{j_2} dx_{j_1}. \tag{62}$$

LEMMA 4. *For all $m \geq m^1$, any $\mathbf{k} \in \mathcal{K}$ with $\sum_\theta k_\theta = k$ and permutation $\varrho \in \mathcal{S}_k$ we have*

$$p(\varrho|\mathbf{k}) = \frac{1}{k!} + \frac{1}{k!} \cdot O(m^{-\frac{1}{8}}). \tag{63}$$

Proof. The premise $|f_\theta(x) - f_\theta(\mathrm{M})| \leq c(x-\mathrm{M})^2$ for $x \in I_m$ permits us to choose $\delta \in O(\varepsilon(m)^2)$ with $\delta \leq \frac{1}{2}$ such that

$$(1-\delta) \cdot f_\theta(\mathrm{M}) \leq f_\theta(x) \leq (1+\delta) \cdot f_\theta(\mathrm{M}) \tag{64}$$

and, equivalently,

$$(1-\delta) \cdot \hat{f}_\theta(\mathrm{M}) \leq \hat{f}_\theta(x) \leq (1+\delta) \cdot \hat{f}_\theta(\mathrm{M}) \tag{65}$$

for all types $1 \leq \theta \leq r$ and all $x \in I_m$. Integrating (64) on I_m yields

$$2\varepsilon(m)(1-\delta) \cdot f_\theta(\mathrm{M}) \leq p_\theta \leq 2\varepsilon(m)(1+\delta) \cdot f_\theta(\mathrm{M}). \tag{66}$$

With these bounds we can conclude from $\hat{f}_\theta(\mathrm{M}) = \frac{f_\theta(\mathrm{M})}{p_\theta}$ that

$$\frac{1-\delta}{2\varepsilon(m)} \leq \frac{1}{2\varepsilon(m)(1+\delta)} \leq \hat{f}_\theta(\mathrm{M}) \leq \frac{1}{2\varepsilon(m)(1-\delta)} \leq \frac{1+2\delta}{2\varepsilon(m)} \tag{67}$$

because $1/(1-\delta) \leq 1 + 2\delta$.

Using $(1-\delta)^k \geq 1 - k\delta$ and $(1+\delta)^k \leq 1 + 2k\delta$ for $k\delta \leq 1$,[31] and noting that the hypercube $[0,1]^k$ can be partitioned into $k!$ polytopes $\{x \in [0,1]^k \colon x_{j_1} \leq x_{j_2} \leq \ldots \leq x_{j_k}\}$

[31] The first statement is easily seen by induction on k. The second follows from

$$(1+\delta)^k = \sum_{j=0}^{k} \binom{k}{j} \delta^j \leq 1 + \sum_{j=1}^{k} \frac{1}{j!} \underbrace{(k\delta)^j}_{\leq k\delta} \leq 1 + k\delta \underbrace{\sum_{j=1}^{k} \tfrac{1}{j!}}_{\leq e-1} \leq 1 + 2k\delta.$$

Since k is asymptotically proportional to $m^{\frac{5}{8}}$ and $\varepsilon(m)^2 = m^{-\frac{6}{8}}$ we can choose $\delta \in O(m^{-\frac{6}{8}})$ with $(k\delta)^j \leq k\delta$ for $j \geq 1$ whenever m is large enough.

with equal volume, inequality (65) yields

$$
\begin{aligned}
p(\varrho|\mathbf{k}) &\geq (1-\delta)^k \int_{-\varepsilon(m)}^{\varepsilon(m)} \int_{x_{j_1}}^{\varepsilon(m)} \dots \int_{x_{j_{k-1}}}^{\varepsilon(m)} \hat{f}_{j_1}(\mathrm{M}) \dots \hat{f}_{j_k}(\mathrm{M})\, dx_{j_k} \dots dx_{j_2} dx_{j_1} && (68)\\
&= \frac{(1-\delta)^k}{k!} \cdot \hat{f}_{j_1}(\mathrm{M}) \dots \hat{f}_{j_k}(\mathrm{M}) \int_{-\varepsilon(m)}^{\varepsilon(m)} \int_{-\varepsilon(m)}^{\varepsilon(m)} \dots \int_{-\varepsilon(m)}^{\varepsilon(m)} 1\, dx_{j_k} \dots dx_{j_2} dx_{j_1} && (69)\\
&= \frac{(1-\delta)^k}{k!} \cdot \hat{f}_{j_1}(\mathrm{M}) \dots \hat{f}_{j_k}(\mathrm{M}) \cdot (2\varepsilon(m))^k && (70)\\
&\overset{(67)}{\geq} \frac{(1-\delta)^{2k}}{k!} \geq \frac{1-2k\delta}{k!} && (71)
\end{aligned}
$$

and, analogously,

$$
\begin{aligned}
p(\varrho|\mathbf{k}) &\leq (1+\delta)^k \int_{-\varepsilon(m)}^{\varepsilon(m)} \int_{x_{j_1}}^{\varepsilon(m)} \dots \int_{x_{j_{k-1}}}^{\varepsilon(m)} \hat{f}_{j_1}(\mathrm{M}) \dots \hat{f}_{j_k}(\mathrm{M})\, dx_{j_k} \dots dx_{j_2} dx_{j_1} && (72)\\
&= \frac{(1+\delta)^k}{k!} \cdot \hat{f}_{j_1}(\mathrm{M}) \dots \hat{f}_{j_k}(\mathrm{M}) \cdot (2\varepsilon(m))^k && (73)\\
&\overset{(67)}{\leq} \frac{(1+\delta)^k (1+2\delta)^k}{k!} \leq \frac{(1+2\delta)^{2k}}{k!} \leq \frac{1+8k\delta}{k!}. && (74)
\end{aligned}
$$

This implies

$$\left| p(\varrho|\mathbf{k}) - \frac{1}{k!} \right| \leq \frac{8k\delta}{k!}. \tag{75}$$

Because $k \in O(m^{\frac{5}{8}})$ and $\delta \in O(m^{-\frac{6}{8}})$, the relative error $|p(\varrho|\mathbf{k}) - (k!)^{-1}| \big/ (k!)^{-1}$ tends to zero at least as fast as $O(m^{-\frac{1}{8}})$. □

So even though the probabilities of the orderings $\varrho \in \mathcal{S}_k$ of the k agents inside I_m differ depending on which specific ϱ is considered and what are the involved representative types (i.e., which $\mathbf{k}$ is considered), these differences vanish and all orderings become equiprobable as m gets large.

Type θ's conditional pivot probability can be written as

$$\pi^\theta(\mathcal{R}^m|\mathcal{K}) = \sum_{\mathbf{k}\in\mathcal{K}} P(\mathbf{k}) \cdot \Big\{ \sum_{\varrho\in\mathcal{S}_k \,:\, \psi(\mathbf{k},\varrho)=\theta} p(\varrho|\mathbf{k}) \Big\}, \tag{76}$$

where $P(\mathbf{k})$ denotes the probability of $\mathbf{k}$ conditional on event $\{\mathbf{k} \in \mathcal{K}\}$ and function $\psi\colon \mathcal{K} \times \mathcal{S}_k \to \{1, \dots, r\}$ identifies the type θ' of the pivotal member in $\mathcal{R}^m$ when $\mathbf{k}$ describes how the representative types are divided between I_m and its left or right, and ϱ captures the ordering inside I_m. Lemma 4 approximates the probability of ordering ϱ conditional on $\mathbf{k}$ as $1/k!$, and one thus obtains

$$\pi^\theta(\mathcal{R}^m|\mathcal{K}) = \sum_{\mathbf{k}\in\mathcal{K}} P(\mathbf{k}) \cdot \phi_\theta(\mathbf{k}) + O(m^{-\frac{1}{8}}) \tag{77}$$

with

$$\phi_\theta(\mathbf{k}) = \sum_{\varrho \in \mathcal{S}_k \,:\, \psi(\mathbf{k},\varrho)=\theta} \frac{1}{k!}. \tag{78}$$

Because a constant factor $\frac{1}{k!}$ pertains to each ordering $\varrho \in \mathcal{S}_k$, $\phi_\theta(\mathbf{k})$ equals the probability that, as the weights $w_1, w_2, \ldots, w_k$ of the k representatives inside I_m are accumulated in *uniform* random order, the threshold $q(\mathbf{k}) \equiv q^m - \sum_{\theta \in \{1,\ldots,r\}} \triangleleft k_\theta w_\theta$ is first reached by the weight of a type-θ representative. The term $\phi_\theta(\mathbf{k})$ is, therefore, simply the aggregated *Shapley value* of the type-θ representatives in the weighted voting game defined by quota $q(\mathbf{k})$ and weight vector $(w_1, w_2, \ldots, w_k)$. Equation (77) states that $\pi^\theta(\mathcal{R}^m|\mathcal{K})$ converges to the expectation of this Shapley value $\phi_\theta(\mathbf{k})$.

Step 4: Type θ's Shapley value $\phi_\theta(\mathbf{k})$ converges to θ's relative weight in I_m

Condition $\mathbf{k} \in \mathcal{K}$ implies $\frac{1}{3} \cdot \sum_{\theta \in \{1,\ldots,r\}} k_\theta w_\theta \leq q(\mathbf{k}) \leq \frac{2}{3} \cdot \sum_{\theta \in \{1,\ldots,r\}} k_\theta w_\theta$ (see Lemma 3). And our premises guarantee that the relative weight of each individual representative in I_m shrinks to zero. The "Main Theorem*" in Neyman (1982), therefore, has the following corollary:

Lemma 5 (Neyman 1982). *Given that* $\mathbf{k} \in \mathcal{K}$,

$$\phi_\theta(\mathbf{k}) = \frac{k_\theta w_\theta}{\sum_{\theta'=1}^{r} k_{\theta'} w_{\theta'}} \cdot (1 + \mu(m)) \quad \textit{with} \quad \lim_{m \to \infty} |\mu(m)| = 0. \tag{79}$$

Proof. Neyman's theorem considers an infinite sequence of weighted voting games $[q^n; \boldsymbol{w}^n]$ with n voters whose individual relative weights w_i^n approach 0, and in which the relative quota q^n is bounded away from 0 and 100% (or at least $\lim_{n\to\infty} q^n / (\max_i w_i^n) = \infty$). Neyman establishes that[32]

$$\lim_{n \to \infty} |\phi_{T_n}(q^n; \boldsymbol{w}^n) - \sum_{i \in T_n} w_i^n| = 0 \tag{80}$$

holds for any sequence of voter subsets $T_n \subseteq \{1, \ldots, n\}$, where $\phi_{T_n}(q^n; \boldsymbol{w}^n)$ denotes their aggregate Shapley value. (We here consider $q^n = q(\mathbf{k})/w_\Sigma$, $\boldsymbol{w}^n = (w_1, w_2, \ldots, w_k)/w_\Sigma$ and $T_n = \{i \in N \colon \tau(i) = \theta\}$ for $N = \{1, \ldots, k\}$ and $w_\Sigma = \sum_{i \in N} w_i$.[33])

It is trivial that (79) holds if $w_\theta = 0 = \phi_\theta(\mathbf{k})$. So we can assume $w_\theta > 0$, and because there is at least the proportion $\beta > 0$ of representatives from each type in I_m for large m, the aggregate relative weight of θ-type representatives in I_m is bounded away from

[32] We somewhat specialize his finding and adapt the notation.

[33] Our notation leaves some inessential technicalities implicit: $\mathcal{K}$ really refers to a family of such sets, parameterized by m; we implicitly consider a sequence of $\mathbf{k}$-vectors such that $n = k \to \infty$ as $m \to \infty$.

0, i.e.,[34]

$$\liminf_{m\to\infty} \frac{k_\theta w_\theta}{\sum_{\theta'=1}^r k_{\theta'} w_{\theta'}} > 0. \tag{81}$$

Therefore, not only the absolute error $\tilde{\mu}(m)$ made in approximating $\phi_\theta(\mathbf{k}) = \phi_{T_n}(q^n; \boldsymbol{w}^n)$ by $\frac{k_\theta w_\theta}{\sum_{\theta'=1}^r k_{\theta'} w_{\theta'}}$ but also the relative error $\mu(m) \equiv \tilde{\mu}(m)/\frac{k_\theta w_\theta}{\sum_{\theta'=1}^r k_{\theta'} w_{\theta'}}$ must vanish as $m \to \infty$. □

Step 5: Attributing aggregate pivot probabilities to individual representatives

It remains to disaggregate the pivot probabilities $\pi^\theta(\mathcal{R}^m)$ and $\pi^{\theta'}(\mathcal{R}^m)$ of types θ and θ' to individual representatives i and j. The aggregate relative weight of type-θ representatives in the essential interval satisfies

$$\frac{k_\theta w_\theta}{\sum_{\theta'=1}^r k_{\theta'} w_{\theta'}} = \frac{\beta_\theta(m) m p_\theta w_\theta (1 + O(m^{-\frac{2}{5}}))}{\sum_{\theta'=1}^r \beta_{\theta'}(m) m p_{\theta'} w_{\theta'} (1 - O(m^{-\frac{2}{5}}))} = \frac{\beta_\theta(m) p_\theta w_\theta}{\sum_{\theta'=1}^r \beta_{\theta'}(m) p_{\theta'} w_{\theta'}} \left(1 + O(m^{-\frac{2}{5}})\right) \tag{82}$$

for any $\mathbf{k} \in \mathcal{K}$ (see (II) in Lemma 2).[35] Combining this with equations (61), (77) and (79) yields

$$\lim_{m\to\infty} \frac{\pi^\theta(\mathcal{R}^m)}{\pi^{\theta'}(\mathcal{R}^m)} = \lim_{m\to\infty} \frac{\beta_\theta(m) p_\theta w_\theta}{\beta_{\theta'}(m) p_{\theta'} w_{\theta'}} = \lim_{m\to\infty} \frac{\beta_\theta(m) f_\theta(\mathrm{M}) w_\theta}{\beta_{\theta'}(m) f_{\theta'}(\mathrm{M}) w_{\theta'}} \tag{83}$$

for arbitrary $\theta, \theta' \in \{1, \ldots, r\}$. Here, the final equality uses

$$\lim_{m\to\infty} \frac{p_\theta}{p_{\theta'}} = \lim_{m\to\infty} \frac{\int_{-\varepsilon(m)}^{\varepsilon(m)} f_\theta(x) dx}{\int_{-\varepsilon(m)}^{\varepsilon(m)} f_{\theta'}(x) dx} = \frac{f_\theta(\mathrm{M})}{f_{\theta'}(\mathrm{M})}, \tag{84}$$

which can be deduced from (66).

Our main claim then follows from noting that the $m_\theta = \beta_\theta(m) \cdot m$ representatives of type θ in assembly $\mathcal{R}^m$ are symmetric to each other and, therefore, must have identical pivot probabilities in $\mathcal{R}^m$. Hence

$$\lim_{m\to\infty} \frac{\pi_i(\mathcal{R}^m)}{\pi_j(\mathcal{R}^m)} = \lim_{m\to\infty} \frac{\pi^{\tau(i)}(\mathcal{R}^m)/\beta_{\tau(i)}(m)}{\pi^{\tau(j)}(\mathcal{R}^m)/\beta_{\tau(j)}(m)} = \frac{f_i(\mathrm{M}) w_i}{f_j(\mathrm{M}) w_j}. \tag{85}$$

1.C Remarks

Let us end Appendix 1 with remarks on possible generalizations. First, the quadratic bound on f_θ's variation in a neighborhood of M could be relaxed by choosing different constants in equations (33) and (47): $t(m_\theta) = m_\theta^{-b_1}$ with $b_1 < \frac{1}{2}$ is all that is needed in order to ensure a vanishing error probability in (47); and $\varepsilon(m) = m^{-b_2}$ with $b_2 < b_1$ in

[34] The limit itself need not exist because our premises do not rule out that, e.g., m_θ is periodic in m.

[35] To see the second equality note that for $y \in (0, \frac{1}{2})$ we have $\frac{1}{1-y} = 1 + y + y^2 + \ldots \leq 1 + 2y = 1 + O(y)$. Similarly, $\frac{1}{1-y} \geq 1 + y = 1 + O(y)$ and so $\frac{1}{1-y} = 1 + O(y)$.

(33) is sufficient for $\varepsilon(m) \gg t(m_\theta)$. Then a local bound $|f_\theta(x) - f_\theta(\mathrm{M})| \leq cx^a$ for $a > \frac{1-b_2}{b_2}$ is sufficient to establish Lemma 4. Requirement $b_2 < b_1 < \frac{1}{2}$ leaves generous room for $a < 2$, but implies $a > 1$.

Second, it is actually sufficient to assume *local continuity* of all f_θ at M, rather than any strengthening of this,[36] if one appeals to an unpublished result by Abraham Neyman (personal communication). When, as in our setting, all voting weights have the same order of magnitude, the uniform convergence theorem of Neyman (1982) for the Shapley value can be generalized to hold for all *random order values* that are 'sufficiently close' to the Shapley value. More specifically, consider the expected marginal contribution of a voter $i \in \{1, \ldots, k\}$

$$\Phi_i(v) \equiv \sum_{\varrho \in \mathcal{S}_k} p(\varrho) \cdot [v(T_i(\varrho) \cup \{i\}) - v(T_i(\varrho))] \tag{86}$$

in a weighted voting game $v = [q; w_1, \ldots, w_k]$, where any given permutation $\varrho \in \mathcal{S}_k$ on $N = \{1, \ldots, k\}$ has probability $p(\varrho)$, and $T_i(\varrho) \subset N$ denotes the set of i's predecessors in ϱ, i.e., $T_i(\varrho) = \{j\colon \varrho(j) < \varrho(i)\}$. The random order value $\Phi(v)$ equals the Shapley value $\phi(v)$ if $p(\varrho) = \frac{1}{k!}$. This equiprobability can, for instance, be obtained by letting ϱ be defined by the order statistics of a vector of random variables $\boldsymbol{X} = (X_1, \ldots, X_k)$ with mutually independent and $[0,1]$-uniformly distributed $X_1, \ldots, X_k$. The latter assumption can be relaxed somewhat without destroying the asymptotic proportionality of i's weight w_i and $\Phi_i(v)$ which Neyman (1982) has established when $\Phi(v) = \phi(v)$:

Theorem 3 (Neyman, personal communication). *Fix $L > 1$. For every $\varepsilon > 0$ there exist $\delta > 0$ and $K > 0$ such that if v is the weighted voting game $v = [q; w_1, \ldots, w_k]$ with $w_1, \ldots, w_k > 0$, $\sum_{i=1}^k w_i = 1$, $K \cdot \max_i w_i < q < 1 - K \cdot \max_i w_i$, $\max_{i,j} w_i/w_j < L$, and $\{p(\varrho)\}_{\varrho \in \mathcal{S}_k}$ in (86) is defined by the order statistics of independent $[0,1]$-valued random variables $X_1, \ldots, X_k$ with densities f_i such that $1 - \delta < f_i(x) < 1 + \delta$ for every $x \in [0,1]$ and $i \in \{1, \ldots, k\}$ then*

$$\sum_{i=1}^{k} |w_i - \Phi_i(v)| < \varepsilon. \tag{87}$$

Of course, one can equivalently let $\{p(\varrho)\}_{\varrho \in \mathcal{S}_k}$ be defined by the order statistics of independent I_m-valued random variables with densities $\hat{f}_1, \ldots, \hat{f}_k$, instead of $[0,1]$-valued ones, if the theorem's condition $1 - \delta < f_i(x) < 1 + \delta$ is replaced by the requirement that $\frac{1-\delta}{2\varepsilon(m)} < \hat{f}_i(x) < \frac{1+\delta}{2\varepsilon(m)}$ for all $x \in I_m$.

[36] Local continuity of f_θ is obviously necessary: a modification of $f_\theta(\mathrm{M})$ – with $f_\theta(x)$ unchanged for $x \neq \mathrm{M}$ – would affect $w_i f_\theta(\mathrm{M})$ but not $\pi_i(\mathcal{R}^m)$. Also the requirement of *positive density* at the common median cannot be relaxed. This is seen, e.g., by considering densities f_i, f_j where $f_i(x) = 0$ on a neighborhood $N_\varepsilon(\mathrm{M})$ while $f_j(\mathrm{M}) = 0$ with $f_j(x) > 0$ for $x \in N_\varepsilon(\mathrm{M}) \setminus \{\mathrm{M}\}$; then $\pi_i(\mathcal{R}^m)/\pi_j(\mathcal{R}^m)$ converges to 0 rather than w_i/w_j.

The values of δ and L which one obtains for a given ε in Theorem 3 apply to *any* value of k. We consider the weighted voting subgames played by the $k = \sum_{\theta\in\{1,\ldots,r\}} k_\theta$ representatives with realizations $\lambda_i \in I_m$ for given $\mathbf{k} \in \mathcal{K}$. The relative weight of any such representative i, $\hat{w}_i = w_i / \sum_{\theta\in\{1,\ldots,r\}} k_\theta w_\theta$, approaches zero as $m \to \infty$; and so does the maximum relative weight. Recalling that the corresponding subgame's relative quota $\hat{q} = q(\boldsymbol{k}) / \sum_{\theta\in\{1,\ldots,r\}} k_\theta w_\theta$ is bounded by $\frac{1}{3} \leq \hat{q} \leq \frac{2}{3}$, the condition $K \cdot \max_i \hat{w}_i < \hat{q} < 1 - K \cdot \max_i \hat{w}_i$ is satisfied when m is sufficiently large. Any null players with $w_i = 0$ can w.l.o.g. be removed from consideration. Then all weights have the same order of magnitude, i.e., the choice of L such that $\max_{i,j} \hat{w}_i/\hat{w}_j < L$ holds for all $\mathbf{k} \in \mathcal{K}$ is trivial.

Moreover, the conditional densities $\hat{f}_\theta$ in our setup satisfy $\frac{1-\delta}{2\varepsilon(m)} < \hat{f}_i(x) < \frac{1+\delta}{2\varepsilon(m)}$ for every $\theta \in \{1,\ldots,r\}$ and $x \in I_m$ when m is large enough. Specifically, continuity of f_θ in a neighborhood of M implies that for any given $\varepsilon > 0$ there exists $\Delta(\varepsilon) > 0$ with $\lim_{\varepsilon\downarrow 0} \Delta(\varepsilon) = 0$ such that

$$(1 - \Delta(\varepsilon)) \cdot f_\theta(\mathrm{M}) \leq f_\theta(x) \leq (1 + \Delta(\varepsilon)) \cdot f_\theta(\mathrm{M}) \tag{88}$$

for all $x \in [\mathrm{M} - \varepsilon, \mathrm{M} + \varepsilon]$ and all $\theta \in \{1,\ldots,r\}$ (cf. inequality (29)). Similarly to inequality (37) we then conclude

$$(1 - \Delta(\varepsilon)) f_\theta(\mathrm{M}) \cdot 2\varepsilon \leq p_\theta \leq (1 + \Delta(\varepsilon)) f_\theta(\mathrm{M}) \cdot 2\varepsilon. \tag{89}$$

Combining the last two inequalities with inequality (45) yields

$$\frac{(1 - \Delta(\varepsilon))}{(1 + \Delta(\varepsilon)) \cdot 2\varepsilon} \leq \hat{f}_\theta(x) \leq \frac{(1 + \Delta(\varepsilon))}{(1 - \Delta(\varepsilon)) \cdot 2\varepsilon}. \tag{90}$$

So considering $\varepsilon = \varepsilon(m)$ and any fixed δ, the conditional densities $\hat{f}_\theta$ satisfy $\frac{1-\delta}{2\varepsilon(m)} < \hat{f}_i(x) < \frac{1+\delta}{2\varepsilon(m)}$ for every $\theta \in \{1,\ldots,r\}$ and $x \in I_m$ when m is sufficiently large.

Hence, all premises in Neyman's unpublished Theorem 3 are satisfied by the corresponding weighted voting subgames of agents with ideal points in I_m. Theorem 3, therefore, ensures the approximate weight proportionality of the aggregate random order value Φ of the type-θ representatives. Now if one recalls (76) and notices that the bracketed sum equals $\Phi(v)$ with $v = [\hat{q}; \hat{w}_{j_1}, \ldots, \hat{w}_{j_k}]$ when $j_1, \ldots, j_k$ denote the representatives with ideal points in I_m, we can replace Lemmata 4–5 by the following:

Lemma 6.

$$\pi^\theta(\mathcal{R}^m|\mathcal{K}) = \frac{k_\theta w_\theta}{\sum_{\theta'=1}^{r} k_{\theta'} w_{\theta'}} \cdot (1 + \mu(m)) \quad \textit{with} \quad \lim_{m\to\infty} |\mu(m)| = 0. \tag{91}$$

The proof of Theorem 1 can then be concluded by appealing to (61), hence

$$\lim_{m\to\infty} \frac{\pi^{\theta}(\mathcal{R}^m|\mathcal{K})}{\pi^{\theta'}(\mathcal{R}^m|\mathcal{K})} = \lim_{m\to\infty} \frac{\pi^{\theta}(\mathcal{R}^m)}{\pi^{\theta'}(\mathcal{R}^m)}, \tag{92}$$

and equations (83)–(85). Importantly, the presumption $|f_\theta(x) - f_\theta(\mathrm{M})| \leq c(x-\mathrm{M})^2$ for $x \in [\mathrm{M}-\varepsilon_1, \mathrm{M}+\varepsilon_1]$, which Lemma 4 required, is *not* needed by Lemma 6. It can hence be replaced in Theorem 1 by the simpler requirement that each f_θ is continuous in a neighborhood of M.

Finally, the assumption that only a finite number of different densities and weights are involved in the chain $\mathcal{R}^1 \subset \mathcal{R}^2 \subset \mathcal{R}^3 \subset \ldots$ could be relaxed. However, it is critical that each representative's relative weight vanishes as $m \to \infty$ in order to apply Neyman's results; the asymptotic relation (13) fails to hold, for instance, for a chain with $w_1 = \sum_{j>1} w_j$. And because our result depends on a vanishing *relative* error, which is considered neither by Neyman (1982) nor Theorem 3,[37] it is similarly important that the aggregate relative weight of each type of representatives is bounded away from zero. For instance, with just one representative having weight $w_1 = 1$ and $\beta_2(m) = m-1$ ones with $w_2 = 2$ (see fn. 16), $\lim_{m\to\infty} \pi_1(\mathcal{R}^m) = \lim_{m\to\infty} \pi_j(\mathcal{R}^m) = 0$ for any $j \neq 1$ but the limit of $\pi_1(\mathcal{R}^m)/\pi_j(\mathcal{R}^m)$ may fail to exist.

2. Proof of Theorem 2

The result easily follows from the definition of the Shapley value and the fact that the orderings which are induced by the realizations of the vectors $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$ will coincide with a probability that tends to 1 as t approaches infinity. To see the latter, ignore any null events in which several ideal points or constituency shocks coincide and let $\hat{\varrho}(\mathbf{x})$ denote the permutation of $\{1, \ldots, m\}$ such that $x_i < x_j$ whenever $\hat{\varrho}(i) < \hat{\varrho}(j)$ for the real-valued vector $\mathbf{x} = (x_i)_{i \in \{1,\ldots,m\}}$. We then have:

Lemma 7. *For $i \in \{1, \ldots, m\}$ and $t > 0$ let $\lambda_i^t \equiv t \cdot \mu_i + \tilde{\varepsilon}_i$, where $\mu_1, \ldots, \mu_m$ and $\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_m$ are all mutually independent random variables, $\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_m$ have finite means and variances, and $\mu_1, \ldots, \mu_m$ have an identical bounded density. Then*

$$\lim_{t\to\infty} \Pr(\hat{\varrho}(\boldsymbol{\lambda}^t) = \varrho) = \lim_{t\to\infty} \Pr(\hat{\varrho}(\boldsymbol{\mu}) = \varrho) = \frac{1}{m!} \tag{93}$$

for each permutation ϱ of $\{1, \ldots, m\}$.

Proof. Let us denote the finite variance of $\tilde{\varepsilon}_i$ by σ_i^2 and let $U \equiv (\max_i |\mathbf{E}[\tilde{\varepsilon}_i]|)^3$. We can choose a real number k such that the bounded density function h of μ_i, with

[37] See, however, Lindner and Machover (2004), where conditions very similar to ours are considered for the Shapley and Banzhaf values, and the related discussion by Lindner and Owen (2007).

$i \in \{1, \dots, m\}$, satisfies $h(x) \le k$ for all $x \in \mathbb{R}$. For any given realization $\mu_j = x$, the probability of the independent random variable μ_i assuming a value inside interval $(x - 4t^{-\frac{2}{3}}, x + 4t^{-\frac{2}{3}})$ is bounded above by $k \cdot 8t^{-\frac{2}{3}}$. We can infer that the event $\left\{|\mu_i - \mu_j| < 4t^{-\frac{2}{3}}\right\}$, which is equivalent to the event $\left\{|t\mu_i - t\mu_j| < 4t^{\frac{1}{3}}\right\}$, has a probability of at most $k \cdot 8t^{-\frac{2}{3}}$ for any $i \ne j \in \{1, \dots, m\}$. And we can conclude from Chebyshev's inequality that $\Pr(|\tilde{\epsilon}_i - \mathbf{E}[\tilde{\epsilon}_i]| < t^{\frac{1}{3}})$ is at least $1 - \sigma_i^2 \cdot t^{-\frac{2}{3}}$. For $t \ge U$, we have $|\mathbf{E}[\tilde{\epsilon}_i]| \le t^{\frac{1}{3}}$; and if $|\tilde{\epsilon}_i - \mathbf{E}[\tilde{\epsilon}_i]| < t^{\frac{1}{3}}$ holds then also

$$2t^{\frac{1}{3}} > |\mathbf{E}[\tilde{\epsilon}_i]| + |\tilde{\epsilon}_i - \mathbf{E}[\tilde{\epsilon}_i]| \ge |\tilde{\epsilon}_i| \tag{94}$$

by the triangle inequality. Hence, the probability for (94) to hold when $t \ge U$ is $\Pr(|\tilde{\epsilon}_i| < 2t^{\frac{1}{3}}) \ge 1 - \sigma_i^2 \cdot t^{-\frac{2}{3}}$ for each $i \in \{1, \dots, m\}$.

Now consider the joint event that (i) $|t\mu_i - t\mu_j| \ge 4t^{\frac{1}{3}}$ for *all* pairs $i \ne j \in \{1, \dots, m\}$ and (ii) that $|\tilde{\epsilon}_i| < 2t^{\frac{1}{3}}$ for *all* $i \in \{1, \dots, m\}$. In this event, the ordering of $\lambda_1^t, \dots, \lambda_m^t$ is determined entirely by the realization of $t\mu_1, \dots, t\mu_m$; in particular, $\hat{\varrho}(\boldsymbol{\lambda}^t) = \hat{\varrho}(\boldsymbol{\mu})$. Using the mutual independence of the considered random variables this joint event must have a probability of at least

$$\prod_{s=1}^{\binom{m}{2}} \left(1 - k \cdot 8t^{-\frac{2}{3}}\right) \cdot \prod_{i=1}^{m} \left(1 - \sigma_i^2 \cdot t^{-\frac{2}{3}}\right) \ge 1 - \left(8k\binom{m}{2} + \sum_{i=1}^{m} \sigma_i^2\right) \cdot t^{-\frac{2}{3}} \tag{95}$$

for $t \ge U$. The right hand side tends to 1 as t approaches infinity. It hence remains to acknowledge that any ordering $\hat{\varrho}(\boldsymbol{\mu})$ has an equal probability of $1/m!$ because $\mu_1, \dots, \mu_m$ are i.i.d. □

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