Monotonicity of Power in Weighted Voting Games with Restricted Communication

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Abstract

Indices that evaluate the distribution of power in simple games are commonly required to be monotonic in voting weights when the game represents a voting body such as a shareholder meeting, parliament, etc. The standard notions of local or global monotonicity are bound to be violated, however, if cooperation is restricted to coalitions that are connected by a communication graph. This paper proposes new monotonicity concepts for power in games with communication structure and investigates the monotonicity properties of the Myerson value, the restricted Banzhaf value, the position value, and the average tree solution.

Keywords: power measurement, weighted voting, restricted communication, monotonicity, centrality, Myerson value, position value, average tree solution

JEL Classification: C71

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1. Introduction

Power is one of the most important concepts in the social sciences, but difficult to quantify. Attempts to measure specific aspects of it in particular contexts – notably voting bodies that take "yes" or "no" decisions – have inspired numerous studies.[1] Commonly, a model of a given decision body is defined simply by a partition of all subsets of players into winning and losing coalitions. But some contexts make it expedient to incorporate additional information. Therefore, many specialized indices have been derived from baseline solutions such as the Shapley or Banzhaf values (Shapley 1953; Banzhaf 1965). For instance, one line of research takes into account that players belong to *a priori unions* which either support or reject proposals as a bloc (pioneered by Aumann and Drèze 1974; Owen 1977). Another one, which is followed here, investigates the assumption that coalitions can only form between players that, in an abstract sense, can "communicate" with another. Namely, agents are presumed to cooperate only if they are connected in a graph that reflects, for example, ideological, social, or spatial proximity. This gives rise to *games with restricted communication* or *communication structure*. Prominent solution concepts for these games include the Myerson value (Myerson 1977) and the position value (Borm et al. 1992).

Identification of which value or power index is particularly satisfactory amongst the many conceivable ones is not easy, even when one restricts attention to situations which can plausibly be modeled as simple games without additional information. In determining whether an index is suitable in a given context (or more suitable than another), the respective axiomatic characterizations, probabilistic foundations, and possible interpretations play an important role. Moreover, the *monotonicity properties* of a power index are commonly regarded as a major criterion. They provide a first test of whether a candidate index fits one's basic intuition about power in the particular context. Specifically, consider a *weighted voting game* $[q; w_1, \ldots, w_n]$ with weights $(w_1, \ldots, w_n)$ and quota q such that a coalition $S \subseteq \{1, \ldots, n\}$ of players is winning if and only if the combined weight of the members of S exceeds or equals q. It seems compelling to require that a power index is *monotonic* in the following sense: if player i has weakly greater *voting weight* than player j, then a plausible index should indicate weakly greater *voting power* for i. Satisfaction of this local monotonicity – and a related *global monotonicity* property that compares a given player's power across games – are by many regarded as a *sine qua non* for sensible power measures (see Felsenthal and Machover 1998, 245f). The default identification of more voting weight with more voting power is problematic, however, whenever additional structure such as procedural rules, a priori unions, or communication restrictions affect agents' decision making. Procedural advantage or a central political position can (over-)compensate low voting weight.

This paper proposes adaptations of the conventional notions of local and global monotonicity for power indices and values for games with restricted communication. We focus on

[1] See Felsenthal and Machover (2006) for a brief historical survey. Comprehensive overviews are given by Felsenthal and Machover (1998) and Laruelle and Valenciano (2008).

binary collective decisions, that is, situations in which any coalition can be classified as either winning or losing, and weighted voting games in order to simplify our presentation.[2] All definitions and results for monotonicities with respect to communication possibilities, however, apply also to general superadditive TU games.

Possible restrictions of communication may reflect a variety of social, hierarchical, legal, or technological constraints on the support which is necessary for a successful motion or project, and possibly on the sequence in which coalitions can form. For example, a linear communication graph could model three political parties that are ordered along a left-right spectrum. Members of the left wing and the right wing parties are not literally unable to speak to each other; they may very well pass a proposal jointly if both prefer it to the status quo. However, a model that includes no link between them reflects the presumption that they will never jointly support a proposal unless the centrist party does so, too. In an abstract sense, they cannot "communicate" directly but only when being connected by the centrists. In such examples, graph-constrained communication is a proxy for cooperation with ideological friction in the tradition of Axelrod (1970). In other contexts, a communication graph might capture actual physical constraints on cooperation (see Ambec and Sprumont 2002 on sharing a river or Curiel et al. 1989 on sequencing situations).[3] Real and virtual social networks have gained enormous scientific attention (see Goyal 2007, Vega-Redondo 2007, and Jackson 2008 for overviews), and attest to implicit or explicit restrictions on cooperation in contexts ranging from group buying to political insurrection coordinated by SMS text messaging.

The straightforward monotonicity requirements alluded to for conventional weighted voting games need to be adapted in such environments.[4] Weight monotonicity should be confined to players and games that, at least, are *comparable* in communication possibilities. Conversely, additional monotonicity requirements with respect to communication possibilities should relate players who have identical weights but can naturally be ordered in other respects. The difficulty is to define notions of monotonicity that are, first, not too restrictive and therefore trivial (e.g., requiring local monotonicity in weight only for players that have absolutely identical communication possibilities). Second, they should not be so permissive as to render all of the established indices non-monotonic (e.g., calling for monotonicity in weight as soon as two players have the same number of links). Finally, it seems desirable that one preserves some of the structural relationships that connect local and global monotonicity properties on other domains (see Turnovec 1998; Alonso-Meijide et al. 2009).

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[2] A weighted voting representation does not presuppose any actual voting. In a procurement context, say, weights w_i might reflect individual demands and quota q can represent the threshold for a given discount.

[3] Van den Brink et al. (2011) point to other examples in which communication graphs usefully structure potential cooperation – *assignment games* (Shapley and Shubik 1972), for instance, in which sellers and buyers have to coordinate on who buys from whom, and games in which banks share ATM networks (Bjorndal et al. 2004).

[4] See Holler and Napel (2004a, 2004b) for a general discussion of monotonicity when additional information complements a standard weighted voting game.

The monotonicity notions that we propose are meant to clarify and compare the behavior of a wide range of indices, rather than to axiomatically characterize specific ones. For illustration, we consider the Myerson value, the restricted Banzhaf value (Owen 1986), the position value, and the average tree solution (Herings et al. 2008; 2010), which have received the greatest attention in the literature on games with communication structure. Instructive examples of how particular notions of monotonicity can be employed in axiomatic characterizations are provided by Allingham (1975), Young (1985) or, for games with communication structure, Hamiache (2011). For an in the general case too demanding global monotonicity requirement (that a new link never harms *any* player), Slikker (2005b) has investigated the related question of which restriction of the considered class of games guarantees that a given solution concept behaves monotonically.

In section 2 we introduce our notation and basic definitions. Section 3 contains descriptions of the mentioned solution concepts for games with restricted communication. In section 4 we devise and investigate monotonicity requirements concerning the weights of players, and in section 5 we formalize the intuitive requirement that greater power be indicated for players with "better connections", i.e., superior communication possibilities. We comment on relations to the measurement of *centrality* in social networks and conclude in section 6.

2. Preliminaries

2.1. **Weighted Voting Games.** A (monotone) *simple game* is a pair (N, v) where $N = \{1, \ldots, n\}$ is the non-empty and finite *set of players* and the *characteristic function* $v\colon 2^N \to \{0, 1\}$ defines whether any coalition $S \subseteq N$ is *winning* ($v(S) = 1$) or *losing* ($v(S) = 0$). It is required that (i) the empty coalition $\varnothing$ is losing ($v(\varnothing) = 0$), (ii) the grand coalition N is winning ($v(N) = 1$), and (iii) v is monotone ($S \subseteq T \Rightarrow v(S) \leq v(T)$). A simple game is called *proper* if any two winning coalitions have a non-empty intersection, i.e., $v(S) = 1$ implies $v(N \backslash S) = 0$. We call a winning coalition S a *minimal winning coalition* (MWC) if every proper subcoalition $T \subset S$ is losing. Players who do not belong to any MWC are known as *dummy* or *null players*.

A *weighted voting game* is a simple game that can be represented by a pair $[q; w]$, which consists of *(voting) weights* $w = (w_1, \ldots, w_n)$ and a *quota* $q \leq \sum w_i$ such that $v(S) = 1 \Leftrightarrow \sum_{i \in S} w_i \geq q$. Throughout the paper we assume that $q > \frac{1}{2} \sum_i w_i$, which ensures properness. Not every simple game allows for a representation $[q; w]$, written as $(N, v) = [q; w]$, while those which do have many equivalent ones.[5] We denote the set of all weighted voting games by $\mathcal{W}$.

A *power index* f is a mapping that assigns an n-dimensional real-valued vector $f(N, v) = (f_1(N, v), \ldots, f_n(N, v))$ to each simple game (N, v), where $f_i(N, v)$ is interpreted as player i's power in game (N, v). The two most prominent power indices are the *Shapley-Shubik index*

[5] Taylor and Zwicker (1999) provide characterizations of those simple games which are weighted voting games.

(*SSI*) (Shapley and Shubik 1954) and the *Banzhaf index* (*BI*) (Banzhaf 1965) defined by

$$SSI_i(N,v) \equiv \sum_{S\subseteq N\setminus\{i\}} \frac{s!(n-s-1)!}{n!}\,(v(S\cup\{i\})-v(S))\,, \quad i=1,\ldots,n, \tag{1}$$

where s denotes the cardinality of S, and

$$BI_i(N,v) \equiv \frac{1}{2^{n-1}} \sum_{S\subseteq N\setminus\{i\}} (v(S\cup\{i\})-v(S))\,, \quad i=1,\ldots,n. \tag{2}$$

They are restrictions of particular *semivalues* for general TU games – the Shapley value (Shapley 1953) and the Banzhaf value (Owen 1975), respectively – to simple games. Semivalues are weighted averages of a player's marginal contributions to coalitions

$$f_i(N,v) \equiv \sum_{S\subseteq N\setminus\{i\}} p_S\,(v(S\cup\{i\})-v(S))\,, \quad i=1,\ldots,n, \tag{3}$$

where the weights p_S depend on a coalition S only via its cardinality s and define a probability distribution, i.e., $p_S = p_s \geq 0$ with $\sum_{s=0}^{n-1}\binom{n-1}{s}p_s = 1$ (see Weber 1988).

Some properties of power indices are widely considered as desirable on the domain of simple or weighted voting games. For instance, a power index f satisfies the *null player property* if it assigns zero power to null players in any simple game (N,v). Another property pertains to possible symmetries: for given (N,v), players i and j are called *symmetric* if there exists a permutation π on N which (i) maps i to j and (ii) under which v is invariant, i.e., $v(S) = 1 \Leftrightarrow v(\pi(S)) = 1$.[6] In case of weighted voting games, this is equivalent to existence of a representation $(N,v) = [q;w]$ for which $w_i = w_j$. A power index f is called *symmetric* if it assigns equal power $f_i(N,v) = f_j(N,v)$ to symmetric players i and j in any given simple game (N,v). On the domain of weighted voting games, a power index f is said to be *locally monotonic* if for all $(N,v) = [q;w]$ it holds that $w_i \geq w_j$ implies $f_i(N,v) \geq f_j(N,v)$. It is called *globally monotonic* if for all $(N,v) = [q;w]$ and $(N,v') = [q;w']$ it holds that $f_i(N,v) \geq f_i(N,v')$ if $w_i \geq w'_i$, $w_j \leq w'_j$ for all $j \neq i$, and $\sum_j w_j \geq \sum_j w'_j$.[7] If a power index f is symmetric, global monotonicity of f implies local monotonicity of f. Semivalues, including *SSI* and *BI*, satisfy all four properties.

2.2. **Restricted Communication.** A *simple game with communication structure* is a triplet (N,v,g) where (N,v) is a simple game and $g \subseteq g^N \equiv \{\{i,j\}\,|\,i,j\in N, i\neq j\}$ is an *unweighted and undirected graph* on N. In what follows we pay particular attention to *weighted voting games with communication structure* or *restricted communication*, i.e., those cases where $(N,v)\in\mathcal{W}$. We denote the collection of all such games by $\mathcal{W}^{\mathcal{G}}$. Two players i and j are able to cooperate directly or to *communicate* in (N,v,g) if the *link* $\{i,j\}$ between these two players is a member

[6]Identical marginal contributions of i and j, i.e., $v(S\cup\{i\}) = v(S\cup\{j\})$ for all $S \not\ni i,j$, are sufficient for this but not necessary. Considering (N,v) with MWCs $\{1,2,3\}$, $\{2,3,4\}$ and $\{3,4,5\}$, and permutation $\pi(1,2,3,4,5) = (5,4,3,2,1)$, one sees that 1 and 5 are symmetric. But their marginal contributions, e.g., to coalition $\{2,3\}$ differ.

[7]Global monotonicity is implied by Young's (1985) notion of *strong monotonicity* for TU values.

of graph g. Those players j that i can communicate with are collected in the *set of neighbors* of player i, $N_i(g) = \{j \mid \{i, j\} \in g\}$.

Players $i, j \in S$ are called *connected in S by g* if either $i = j$ or if there exists a *path* in g from i to j which stays in S, i.e., there are players $k_0, \ldots, k_l \in S$ such that $k_0 = i$, $k_l = j$, and $\{k_0, k_1\}, \ldots, \{k_{l-1}, k_l\} \in g$. Coalition S is *connected by g* if all players $i, j \in S$ are connected in S by g. The coalitions involving player i which are connected by g will be collected in the set

$$\mathcal{C}_i(g) \equiv \{S \subseteq N \mid S \text{ is connected on } g \text{ and } i \in S\}.$$

Subset $T \subseteq S$ is a *component* of S in g if it is connected by g and no T' with $T \subset T' \subseteq S$ is connected by g. Thus, g induces a unique partition S/g (say "S divided by g") of any coalition S into its components,

$$S/g \equiv \{\, \{j \mid i \text{ and } j \text{ connected in } S \text{ by } g\} \mid i \in S\}. \tag{4}$$

$S/g = \{S\}$ if and only if S is connected by g. The *full graph* g^N and the *empty graph* $\emptyset$ induce the trivial partitions $S/g^N = \{S\}$ and $S/\emptyset = \{\{i\} \mid i \in S\}$, respectively. Another useful definition is $g|_S$, the restriction of g to coalition S given by

$$g|_S \equiv \{\, \{i, j\} \mid \{i, j\} \in g \text{ and } i, j \in S\}.$$

The assumption that two players i and j can cooperate directly in (N, v, g) only if they can communicate, and hence can cooperate indirectly in a coalition only if that coalition is connected, gives rise to a *restricted game* $(N, v/g)$. Its characteristic function v/g is defined by

$$v/g\,(S) \equiv \sum_{T \in S/g} v(T), \quad S \subseteq N. \tag{5}$$

In other words, any coalition S is split into its components and the worth of S equals the total worth of these subcoalitions.[8] In those cases where no winning coalition of (N, v) is connected by g, in particular also not the grand coalition N, the restricted game $(N, v/g)$ is a *null game* with $v/g \equiv 0$. In all other cases, the restricted game $(N, v/g)$ induced by $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$ is a simple game in which a coalition is winning if and only if it contains a winning component in g. A communication structure can thus be implicit in the specification of a standard simple game. Note that $(N, v/g)$ need *not* be a weighted voting game even though $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$.[9] For the full graph g^N, the restricted game induced by (N, v, g^N) coincides with (N, v).

We call two players i and j *symmetric* in $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$ if, for some representation $(N, v) = [q; w]$, there is a permutation π on N which (i) maps i to j and (ii) leaves weights w and graph g invariant, i.e., for the permuted weights $\pi(w) \equiv (w_{\pi^{-1}(k)})_{k \in N}$ and the permuted graph

[8] In non-proper simple games, several disjoint subcoalitions $T \subset S$ might be winning. Since we have ruled out this case by presuming $q > \sum_i w_i/2$, we could replace $\sum v(T)$ by $\max v(T)$.

[9] Consider, e.g., $(N, v) = [4; 2, 1, 1, 1, 2]$ and $g = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$. Then $(N, v/g)$ has just two MWCs: $\{1, 2, 3\}$ and $\{3, 4, 5\}$. For any representation $[q; w]$, we must have $w_1 + w_2 + w_3 \geq q$ and $w_3 + w_4 + w_5 \geq q$. This implies $w_3 + \max(w_1, w_2) + \max(w_4, w_5) \geq q$ and hence existence of a third MWC – a contradiction.

$\pi(g) \equiv \{\{\pi(k), \pi(l)\} \mid \{k,l\} \in g\}$ we have $\pi(w) = w$ and $\pi(g) = g$.[10] This necessitates, of course, that players i and j have identical weights, i.e., $w_i = w_j$. Two players who are symmetric in a weighted voting game with communication structure are necessarily symmetric in the corresponding restricted game (in the sense of the definition in section 2.1).[11]

A *value* or *power index for (games with) communication structures* or *restricted communication* is a mapping f that assigns an n-dimensional real-valued vector $f(N,v,g) = (f_1(N,v,g), \ldots, f_n(N,v,g))$ to each weighted voting game with communication structure $(N,v,g) \in \mathcal{W}^{\mathcal{G}}$, where $f_i(N,v,g)$ is interpreted as player i's power in game (N,v,g). Such f satisfies *symmetry (SYM)* if $f_i(N,v,g) = f_j(N,v,g)$ whenever players i and j are symmetric in $(N,v,g) \in \mathcal{W}^{\mathcal{G}}$.[12] It satisfies *component efficiency (CE)* if $\sum_{i \in S} f_i(N,v,g) = v(S)$ for each $S \in N/g$ in any given $(N,v,g) \in \mathcal{W}^{\mathcal{G}}$.

3. Selected Indices

3.1. **Restricted Semivalues.** The literature on simple games often explicitly distinguishes between the Banzhaf *value* and the Banzhaf *index*, or the Shapley value and its restriction to simple games, the Shapley-Shubik index. We make no such distinction here and refer to a "value" even when the corresponding mapping is considered on the domain of *simple* games with communication structures. The first value concept that has been proposed and axiomatized specifically for situations with restricted communication is the *Myerson value* MV (Myerson 1977). Used as a power index for games with communication structures (N,v,g), it assigns the Shapley-Shubik index of the corresponding restricted game $(N,v/g)$ to each player, i.e.,

$$MV(N,v,g) \equiv SSI(N,v/g). \tag{6}$$

In the special case of (N,v) being a *unanimity game*, i.e., $v(N) = 1$ and $v(S) = 0$ for all $S \subset N$, MV indicates the power distribution $(1/n, \ldots, 1/n)$ if the grand coalition is connected by g, and $(0, \ldots, 0)$ otherwise. In case of the full graph, $MV(N,v,g^N)$ coincides with $SSI(N,v)$.

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[10]More generally, *symmetry* of players i and j in a *TU game with restricted communication* (N,v,g) requires a permutation π on N which (i) maps i to j and (ii) for which $v(S) = v(\pi(S))$ for all $S \subseteq N$ and $\pi(g) = g$.

[11]Requiring only that two players are symmetric in the restricted game $(N,v/g)$ according to section 2.1's definition would be a strictly weaker notion of symmetry, which "trades off" asymmetries in weight and communication: for $(N,v) = [2;1,0,1]$ and $g = \{\{1,2\},\{2,3\}\}$, player 1 has more weight than 2, but 2 has more links. They are hence not symmetric in (N,v,g), but happen to be in $(N,v/g)$ (where only N is winning).

[12]A value f satisfies *anonymity* if for all $(N,v,g), (N,v',g') \in \mathcal{W}^{\mathcal{G}}$ with $(N,v) = [q;w]$ and $(N,v') = [q;w']$ for which there is a permutation π on N such that $w' = \pi(w)$ and $g' = \pi(g)$ it holds that $f(N,v',g') = \pi(f(N,v,g))$. Anonymity implies symmetry.

Analogously, the *restricted Banzhaf index RBI* (Owen 1986) maps any given game (N, v, g) to the Banzhaf index of the corresponding restricted game $(N, v/g)$,[13] i.e.,

$$RBI(N, v, g) \equiv BI(N, v/g). \quad (7)$$

MV is component efficient, while RBI is not.[14] Symmetry of a baseline value implies symmetry of its restricted version; hence both MV and RBI are symmetric.

3.2. **Position Value.** Operating on restricted game $(N, v/g)$, rather than on (N, v, g) itself, can entail a significant loss of information. Consider, for instance, a large unanimity game and a graph g in which player 1 is the center of a star. Then the restricted game is a unanimity game, too, and all players are symmetric in $(N, v/g)$. Therefore, MV and RBI indicate identical power for all players. But all communication which is necessary in order for the grand coalition to be connected involves player 1. This might plausibly go along with greater power.

The *position value* (Borm et al. 1992) follows this line of reasoning. It measures the power of agents in two steps: first, it considers their links as the "players" in an auxiliary game, the *link game*. The importance of links is picked up by computing their SSI in the link game. Second, the respective two players involved in any of the links share its SSI value equally. More specifically, let (g, v^N) denote the null or simple game played by the links according to the characteristic function v^N given by

$$v^N(h) \equiv v/h(N), \quad h \subseteq g. \quad (8)$$

So, in (N, v, g)'s link game (g, v^N), the worth of a coalition h (of links) is equal to the worth of the grand coalition N (of agents) in the respective restricted game $(N, v/h)$. A coalition h of links, therefore, is winning if and only if it connects a winning coalition of (N, v). The position value PV is then defined by[15]

$$PV_i(N, v, g) \equiv v(\{i\}) + \frac{1}{2} \sum_{\{i,j\} \in g} SSI_{\{i,j\}}(g, v^N), \quad i = 1, \ldots, n. \quad (9)$$

The position value can be viewed as arising from a scenario where communication is established link by link in a random order – until all players cooperate to the highest degree allowed for by the "communication technology" (formalized by g). Players start with their

[13]Any other value for standard (simple) games could in this fashion be adapted to games with communication structure. The list of alternatives includes $(N, v/g)$'s *nucleolus* and also equilibrium payoffs in specific *Baron-Ferejohn bargaining games* derived from $(N, v/g)$ (see, e.g., Le Breton et al. 2012). We here focus on MV and RBI since they have been used in the context of communication structures for longest. For results on the non-monotonicity of the restricted versions of the Deegan-Packel index (Deegan and Packel 1978) and public good index (Holler 1982), see Napel et al. (2011).

[14]See section 3.4 for examples of the violation.

[15]We slightly generalize the definition of Borm et al. (1992) in order to allow for simple games with a dictator. Slikker (2005a) characterizes PV as the only value satisfying CE and *balanced link contributions*, while MV satisfies CE and *balanced (agent) contributions*. Also see van den Brink (2009) for an instructive comparative axiomatization of MV, RBI, PV, and the average tree solution, which is introduced below.

respective stand-alone value $v(\{i\})$ (which is 0, except if i is a dictator); as communication possibilities are activated randomly, one after another, the two players that are connected by any new link share the generated worth $v^N(h \cup \{\{i,j\}\}) - v^N(h)$ equally. Then PV_i is equivalent to player i's expected gains or, in simple games without dictator, half of the probability that one of his links turns a losing coalition into a winning one.[16]

PV satisfies component efficiency and symmetry, but does not reduce to any standard power index for the full graph g^N: in particular, null players are assigned a positive position value in (N, v, g^N) whenever there is no dictator in (N, v).[17] For a unanimity game (N, v) in which N is connected by a cycle-free graph g, a player's position value equals half of his share of links in g.

3.3. **Average Tree Solution.** While the position value is closely related to the Shapley value (considering all orderings of links, rather than agents, as equiprobable), the *average tree solution* of Herings et al. (2008, 2010)[18] takes a different approach. It considers random hierarchies within the communication network in which cooperation spreads from bottom to top. Power is then ascribed to the unique player who turns the coalition of previous, hierarchically lower supporters into a winning one by joining and merging them.

More specifically, for a given graph g on N, a subgraph $t \subseteq g$ is a *spanning tree* on connected coalition S if S is connected by t but not by any $t' \subset t$. A spanning tree thus represents a minimal set of communication links whose activation allows all members of S to cooperate. By its minimality, a spanning tree does not contain links to players outside S. And it cannot contain any *cycle*, i.e., there do not exist distinct players $k_1, \ldots, k_l \in S$, $l \geq 3$, such that $\{k_1, k_2\}, \ldots, \{k_{l-1}, k_l\} \in t$ and $\{k_1, k_l\} \in t$. Therefore, t can be *rooted* at any $j \in S$, which gives links an orientation and results in a *rooted spanning tree* (t, j). A given rooted spanning tree (t, j) reflects a particular order in which the communication links that suffice to bring all members of S together might be activated. The set of neighbors of root player j in (t, j) are naturally called j's *successors*. The neighbors of i's successors, except for i itself, in turn are their successors. Proceeding in this fashion, one inductively obtains the (possibly empty) set of successors $suc_i(t, j) \equiv \{k \mid \{i, k\} \in t \text{ and } i \notin suc_k(t, j)\}$ for all players. $sub_i(t, j)$ denotes the set which contains player i and his *subordinates*, i.e., i's successors, all their successors, and all subsequent successors. If communication links are activated in a given (t, j) from bottom to top, i.e., from terminal players towards the root player j level-by-level, then the *marginal*

[16] Analogously, MV captures expected gains when communication spreads in a lumpy way, i.e., if one random player after another brings in *all* of the communication possibilities which connect him to the players who have already joined. The close relationships between MV and PV have formally been clarified by Casajus (2007) and Kongo (2010).

[17] A star $h = \{\{i, j\} \mid j \neq i\}$ with null player i at its center connects the grand coalition, and is a winning coalition in the link game. Some of i's links therefore have a positive marginal contribution in the link game, which implies $PV_i(N, v, g^N) > 0$ and a violation of the null player property.

[18] It was initially defined for cycle-free graphs and then extended to arbitrary communication structures which connect the grand coalition. While Herings et al. consider one particular class of "admissible" rooted spanning trees, Baron et al. (2011) axiomatize average tree solutions for *any* class of rooted spanning trees.

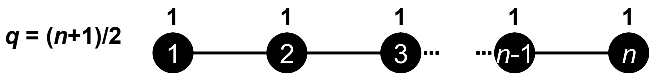

FIGURE 1. Simple majority voting on a left-right scale

contribution of player $i \in S$ in (N, v) with rooted spanning tree (t, j) on S is

$$m_i^v(t, j) \equiv v(sub_i(t, j)) - \sum_{k \in suc_i(t,j)} v(sub_k(t, j)).$$

Thus the marginal contribution is 1 if and only if all coalitions comprising a single successor of i and all that successor's subordinates are individually losing but turn winning when they all become connected by player i joining and merging them.

Given graph g, a rooted spanning tree (t, j) on S is called *admissible* if it holds for any two distinct successors $k, k' \in suc_i(t, j)$ of some player $i \in N$ that the coalition comprising them and their subordinates, $sub_k(t, j) \cup sub_{k'}(t, j)$, is not connected by g. In other words, any two players having the same predecessor are neither directly connected by the original graph g nor indirectly connected via their subordinates.[19] The average tree solution ATS then identifies i's power in (N, v, g) with the average marginal contribution of player i, computed over all admissible rooted spanning trees that can be constructed on the component of N which contains i. Formally, denoting the set of *all admissible* rooted spanning trees of g on S by $T_{S,g}$, the average tree solution is defined by[20]

$$ATS_i(N, v, g) \equiv \frac{1}{|T_{S_i,g}|} \sum_{(t,j) \in T_{S_i,g}} m_i^v(t, j), \quad i = 1, \ldots, n, \tag{10}$$

where $S_i \in N/g$ denotes the component containing i. ATS satisfies component efficiency and symmetry. In case of a unanimity game (N, v) and any graph g which connects the grand coalition, ATS assigns power proportional to the number of spanning trees that can admissibly be rooted in a player; thus, if in addition g is cycle-free, each player is ascribed equal power. In case of the full graph g^N, the average tree solution coincides with the Shapley-Shubik index of (N, v).[21]

3.4. **Illustration.** We conclude the section by comparing the power ascriptions of the above solution concepts in four examples. They involve *simple majority voting on a left-right scale* with $n = 3$, 4, 5, and 6 agents (see figure 1), i.e., $(N, v) = [(n + 1)/2; 1, \ldots, 1]$ and $g =$

[19]One easily verifies that this definition of an admissible rooted spanning tree is equivalent to the one in Herings et al. (2010). Only admissible spanning trees (t, j) guarantee that $m_i^v(t, j)$ equals i's marginal contribution in v/g when i joins his subordinates, i.e., $m_i^v(t, j) = v/g(sub_i(t, j)) - v/g(sub_i(t, j)\setminus\{i\})$. Moreover, Herings et al.'s class of admissible (t, j) is the largest one such that ATS is a *Harsanyi solution* (see Baron et al. 2011).

[20]We slightly generalize the definition of Herings et al. (2010): every component is considered separately in order to allow for graphs that do not connect the grand coalition N. However note that, since we restrict attention to proper games, at most one component is winning.

[21]Given the full graph, any player can have at most one successor in an admissible rooted spanning tree, i.e., any admissible rooted spanning tree is a linear graph. Then $T_{S_i,g}$ represents the set of all orderings on N.

$n = 3$:

i	MV_i	RBI_i	PV_i	ATS_i
1, 3	0.17	0.25	0.25	0.00
2	0.67	0.75	0.50	1.00

$n = 4$:

i	MV_i	RBI_i	PV_i	ATS_i
1, 4	0.08	0.13	0.08	0.00
2, 3	0.42	0.38	0.42	0.50

$n = 5$:

i	MV_i	RBI_i	PV_i	ATS_i
1, 5	0.08	0.13	0.08	0.00
2, 4	0.17	0.25	0.25	0.00
3	0.50	0.50	0.33	1.00

$n = 6$:

i	MV_i	RBI_i	PV_i	ATS_i
1, 6	0.05	0.06	0.04	0.00
2, 5	0.10	0.13	0.13	0.00
3, 4	0.35	0.25	0.33	0.50

Table 1. Power for simple majority voting on a left-right scale with n agents

$\{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}$. Table 1 reports calculations rounded to two decimals. MV and RBI correspond, respectively, to the Shapley-Shubik and Banzhaf indices of the restricted game $(N, v/g)$. The MWCs of this game are all coalitions of $(n+1)/2$ consecutive players if n is odd, or of $n/2+1$ consecutive players if n is even. Hence, MV and RBI are positive but decreasing in distance from the central player(s).[22] The MWCs of the link game (g, v^N), which PV considers, are all coalitions of $(n-1)/2$ consecutive links if n is odd, or of $n/2$ consecutive links if n is even. More central links are critical in more coalitions. Since a link's criticality is shared equally by the involved agents, PV_i is positive but decreases in i's distance from the center. Finally, ATS counts the number of rooted spanning trees in which a given agent is pivotal. For an arbitrary and odd n, the central player is pivotal in every rooted spanning tree. His ATS value, therefore, equals 1, and that of any other player amounts to 0. With an arbitrary even number of players, the left (right) central player is pivotal in all spanning trees rooted in a player on the left (right). This results in an ATS value of 0.5 for both central players, and 0 for all others. ATS is thus the only of the four solution concepts that yields an element of $(N, v/g)$'s *core* and is in line with the intuition of the median voter theorem.

4. Monotonicity with Respect to Weights

We now define two notions of monotonicity for power indices $f\colon \mathcal{W}^{\mathcal{G}} \to \mathbb{R}^n$ which concern the voting weights of players, and then investigate their satisfaction by the considered indices.

4.1. **Definitions.** The first kind of weight monotonicity is *local* in the sense that it concerns comparisons of distinct players in a single given game (N, v, g). Two players i and j will be deemed comparable with respect to weights either when they are symmetric or when the asymmetry between them is only due to different weights – i.e., they would be rendered symmetric by a replacement of w_i and w_j with, e.g., $w'_i = w'_j = \frac{1}{2}(w_i + w_j)$. Both players must in such case have essentially equivalent communication possibilities. We formalize this by

[22]Note that RBI violates component efficiency for $n = 3, 5$, and 6.

calling players i and j *comparable in weights* in game $(N, v, g) = [q; w_1, \ldots, w_n]$ if they are symmetric in some game $(N, v', g) = [q; w'_1, \ldots, w'_n]$ with $w'_k = w_k$ for all $k \neq i, j$ and $w'_i = w'_j$.

Definition **1.** *A power index for communication structures f satisfies* local monotonicity with respect to weights (LW) *or is* LW-monotonic *if for each $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$ with $(N, v) = [q; w_1, \ldots, w_n]$*

$$f_i(N, v, g) \geq f_j(N, v, g)$$

holds for any players i and j that are comparable in weights and for which $w_i \geq w_j$.

An index is thus required to respect a weight advantage for player i over player j, provided that both enjoy equally good communication possibilities.

The second notion of monotonicity concerns the comparison of a single player's power across two different games, and is hence referred to as *global* monotonicity with respect to weights. It demands that whenever the underlying weighted voting game changes in a way that can unambiguously be judged favorable for player i – specifically, i's voting weight increases and/or voting weight is shifted from others to i – then this should have a non-negative effect for player i:

Definition **2.** *A power index for communication structures f satisfies* global monotonicity with respect to weights (GW) *or is* GW-monotonic *if for each $(N, v, g), (N, v', g) \in \mathcal{W}^{\mathcal{G}}$ with $(N, v) = [q; w_1, \ldots, w_n]$ and $(N, v') = [q; w'_1, \ldots, w'_n]$*

$$f_i(N, v, g) \geq f_i(N, v', g)$$

holds whenever $w_i \geq w'_i$, $w_j \leq w'_j$ for all $j \neq i$, and $\sum_j w_j \geq \sum_j w'_j$.

As is the case for local and global monotonicity of power indices for games without communication structure, GW is a stronger requirement than LW in the presence of SYM.

Proposition **1.** *If a power index for communication structures f is GW-monotonic and symmetric, then f is also LW-monotonic.*

Proof. Let f be a power index satisfying GW and SYM, and consider a game $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$ for which LW would require $f_i(N, v, g) \geq f_j(N, v, g)$. Specifically, let $(N, v) = [q; w_1, \ldots, w_n]$ be such that players i and j with $w_i \geq w_j$ are symmetric in $(N, v', g) \in \mathcal{W}^{\mathcal{G}}$ with $(N, v') = [q; w'_1, \ldots, w'_n]$ where $w'_i = w'_j = \frac{1}{2}(w_i + w_j)$ and $w'_k = w_k$ for all $k \neq i, j$. Now, noting that (N, v, g) is more favorable than (N, v', g) for player i in the sense of GW and that the reverse is true for player j, GW and SYM imply

$$f_i(N, v, g) \geq f_i(N, v', g) = f_j(N, v', g) \geq f_j(N, v, g).$$

□

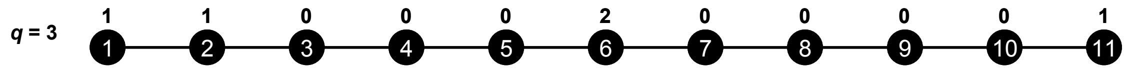

FIGURE 2. Position value violates LW ($PV_2 \ngeq PV_{10}$)

4.2. **Properties of the Considered Indices.** That a baseline index for standard simple games is locally or globally monotonic in weight does not imply LW or GW for its restricted version for games with communication structure.[23] However, we find:

PROPOSITION **2.** *The restricted versions of all semivalues – so, in particular, the Myerson value MV and restricted Banzhaf index RBI – are LW-monotonic and GW-monotonic.*

Proof. Let $f\colon \mathcal{W}^{\mathcal{G}} \to \mathbb{R}^n$ be the restricted version of a semivalue $\tilde{f}$, i.e., $f(N,v,g) \equiv \tilde{f}(N,v/g)$. Now consider two games $(N,v,g),(N,v',g) \in \mathcal{W}^{\mathcal{G}}$ with $(N,v) = [q;w_1,\ldots,w_n]$ and $(N,v') = [q;w'_1,\ldots,w'_n]$ such that $w_i \geq w'_i$, $w_j \leq w'_j$ for all $j \neq i$, and $\sum_k w_k \geq \sum_k w'_k$. For any $S \not\ni i$, choose $T \in (S\cup\{i\})/g$ with $T \ni i$. Then $v/g(T) \geq v'/g(T)$ and $v/g(T\backslash\{i\}) \leq v'/g(T\backslash\{i\})$, and thus

$$v/g(S\cup\{i\}) - v/g(S) = v/g(T) - v/g(T\backslash\{i\}) \geq v'/g(T) - v'/g(T\backslash\{i\}) = v'/g(S\cup\{i\}) - v'/g(S).$$

This and (3) imply $\tilde{f}_i(N,v/g) \geq \tilde{f}_i(N,v'/g)$. Hence $f_i(N,v,g) \geq f_i(N,v',g)$ and f must be GW-monotonic. Because f, as the restricted version of a semivalue, is symmetric, it must also be LW-monotonic (proposition 1). □

In contrast, the position value is *not* LW-monotonic, and hence – being symmetric – *not* GW-monotonic. To see this, consider $(N,v,g) \in \mathcal{W}^{\mathcal{G}}$ with eleven players located on a left-right scale, i.e., $g = \{\{1,2\},\{2,3\},\ldots,\{10,11\}\}$, and $(N,v) = [3;1,1,0,0,0,2,0,0,0,0,1]$ (see figure 2). Players 2 and 10 would be symmetric if w_2 were lowered to $w_{10} = 0$, and hence LW requires $PV_2(N,v,g) \geq PV_{10}(N,v,g)$. The corresponding link game (g,v^N) has two disjoint minimal winning coalitions: one comprising the four links on the left side of the central player 6, $h_1 = \{\{2,3\},\ldots\{5,6\}\}$, and the five links on 6's right, $h_2 = \{\{6,7\},\ldots\{10,11\}\}$. In particular, g's left-most link $\{1,2\}$ is never necessary for the establishment of a winning coalition, and hence it is a null player in (g,v^N): player 2's weight of $w_2 = 1$ makes the inclusion of player 1 redundant. In contrast, $w_{10} = 0$ requires activation of link $\{10,11\}$ before a losing coalition on the right can become winning. The *SSI* in the link game is then 0 for the null link $\{1,2\}$, $\frac{5}{36}$ for any other link $l \in h_1$ on the left side of player 6, and $\frac{4}{45}$ for any link $l \in h_2$. This results in position values $PV_2(N,v,g) = \frac{5}{72} < \frac{4}{45} = PV_{10}(N,v,g)$ – a violation of LW.

The average tree solution satisfies both weight monotonicity requirements:[24]

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[23] To see this, consider $f(N,v,g) \equiv \tilde{f}(N,v/g)$ and the weighted voting game $(N,v) = [4;1,1,0,2,1,1,1]$ with $g = \{\{1,2\},\{2,3\},\ldots,\{6,7\}\}$. The corresponding restricted game has $\{1,2,3,4\}$ and $\{4,5,6\}$ as its MWCs, and hence no representation $[q;w]$. Let the baseline index $\tilde{f}$ be the *SSI* for all simple games apart from this specific $(N,v/g)$, for which we assume $\tilde{f}(N,v/g) = (1/6,1/6,1/6,1/2,0,0,0)$. Because $(N,v/g)$ is no weighted voting game, $\tilde{f}$ *is* locally and globally monotonic. However, f violates LW since it assigns strictly more power to 3 than to 5. It also violates GW since an increase of weight of 1 for player 3 to $(N,v') = [4;1,1,1,2,1,1,1]$ would result in $f(N,v',g) = (0,2/15,2/15,7/15,2/15,2/15,0)$.

[24] This proposition holds for average tree solutions based on any class of rooted spanning trees.

Proposition 3. *The average tree solution ATS is LW-monotonic and GW-monotonic.*

Proof. Consider two games $(N, v, g), (N, v', g) \in \mathcal{W}^{\mathcal{G}}$ with $(N, v) = [q; w_1, \ldots, w_n]$ and $(N, v') = [q; w'_1, \ldots, w'_n]$ such that $w_i \geq w'_i$, $w_j \leq w'_j$ for all $j \neq i$, and $\sum_k w_k \geq \sum_k w'_k$. Let $S_i \in N/g$ be the component containing player i. Then, for any arbitrary rooted spanning tree (t, j) on S_i, we have $v(sub_i(t, j)) \geq v'(sub_i(t, j))$ because $i \in sub_i(t, j)$. Moreover, for all $k \in suc_i(t, j)$, $v(sub_k(t, j)) \leq v'(sub_k(t, j))$ since $i \notin sub_k(t, j)$. Thus $m_i^v(t, j) \geq m_i^{v'}(t, j)$ for any rooted spanning tree (t, j) on S_i, and therefore $ATS_i(N, v, g) \geq ATS_i(N, v', g)$. So ATS must be GW-monotonic and – applying proposition 1 – LW-monotonic, too. □

5. Monotonicity with Respect to Communication Possibilities

We now turn to local and global notions of monotonicity which relate the communication possibilities of players. The latter may be judged as better for player i than for j (or better for a given player in graph g than in g') for different reasons. We highlight one particular local monotonicity requirement and two global ones, and we mention some alternatives. The following definitions do not involve any explicit voting weights. They can, therefore, be extended to situations where a value f is applied to general TU games with communication structure. It is straightforward to check that all monotonicity statements for the mentioned solution concepts in section 5.2 remain valid if the TU games are superadditive.

5.1. **Definitions.** We start again by comparing power indications locally, i.e., for different players in the same game (N, v, g). In analogy to the local monotonicity for weights, we consider two players comparable either when they are symmetric or when an asymmetry solely comes from communication possibilities that one player has in advance. We say that players i and j are *comparable in communication possibilities* in game (N, v, g) whenever i and j are symmetric in some game (N, v, g') where $g' \subseteq g$ and either $g'|_{N\setminus\{i\}} = g|_{N\setminus\{i\}}$ or $g'|_{N\setminus\{j\}} = g|_{N\setminus\{j\}}$. This is a formal way of saying that players i and j would be rendered symmetric if one player's communication possibilities were suitably reduced, i.e., if either some of i's or some of j's links were deleted.[25]

Definition 3. *A power index for communication structures f satisfies* local monotonicity with respect to communication possibilities (LC) *or is* LC-monotonic *if for each* $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$

$$f_i(N, v, g) \geq f_j(N, v, g)$$

holds for any players i and j that are comparable in communication possibilities and for which $|N_i(g)| \geq |N_j(g)|$.

A first notion of global monotonicity with respect to communication possibilities pertains to games (N, v, g) and (N, v, g') where $g \supseteq g'$ and all additional links in g involve player i

[25]It is important to reduce only *one* player's communication possibilities in order to ensure comparability. Otherwise, any i and j with $w_i = w_j$ would satisfy this weak symmetry condition (simply delete all of i's and j's links) so that, in contrast to the formulation above, g would play no role whatsoever.

(i.e., $g|_{N\setminus\{i\}} = g'|_{N\setminus\{i\}}$). One might plausibly require that i's power is weakly greater for communication structure g, since the additional links improve i's communication possibilities in absolute terms (but note that i's new neighbors also have improved possibilities). A complementary second notion looks at games with graphs $g \subseteq g'$ in which the links that are missing in g do not affect the set of connected coalitions involving i, i.e., where $\mathcal{C}_i(g) = \mathcal{C}_i(g')$. Player i's direct and indirect cooperation possibilities are then identical in both graphs, and i's power may be required to be greater for communication structure g than for $g' \supseteq g$ because the removed links improve i's communication possibilities in relative terms: other players have lost some of their cooperation possibilities, while i has not. These two notions can be formalized as follows:

Definition **4.**
(i) A power index for communication structures f satisfies global monotonicity with respect to added communication possibilities (GC$^+$) *or is* GC$^+$-monotonic *if for all* $(N, v, g), (N, v, g') \in \mathcal{W}^{\mathcal{G}}$,

$$f_i(N, v, g) \geq f_i(N, v, g')$$

holds whenever $g \supseteq g'$ *and* $g|_{N\setminus\{i\}} = g'|_{N\setminus\{i\}}$.
(ii) The index satisfies global monotonicity with respect to removed communication possibilities (GC$^-$) *or is* GC$^-$-monotonic *if for all* $(N, v, g), (N, v, g') \in \mathcal{W}^{\mathcal{G}}$,

$$f_i(N, v, g) \geq f_i(N, v, g')$$

holds whenever $g \subseteq g'$ *and* $\mathcal{C}_i(g) = \mathcal{C}_i(g')$.

It is straightforward to check that GC$^+$ is equivalent to the *stability condition* of Myerson (1977), which calls for

$$f_i(N, v, g) \geq f_i(N, v, g\setminus\{i, j\}) \text{ and } f_j(N, v, g) \geq f_j(N, v, g\setminus\{i, j\})$$

whenever $\{i, j\} \in g$. The requirement $\mathcal{C}_i(g) = \mathcal{C}_i(g')$ in GC$^-$ necessitates $N_i(g) = N_i(g')$. The latter is also sufficient for the former if only communication possibilities that connect neighbors of i are lost, i.e., if $\{j, k\} \in g'\setminus g$ implies $j, k \in N_i(g)$.[26, 27]

An analogue to proposition 1, which related global and local monotonicity with respect to weights, does *not* hold for SYM, GC$^+$ and GC$^-$, and LC. That is, a symmetric power index for communication structures may satisfy both GC$^+$ and GC$^-$ but violate LC. To see this, consider figure 3. It defines an anonymous (and hence symmetric) power index for $(N, v) = [2; 1, 1, 0, 0]$ and all possible communication structures g by listing the power vectors

[26]Strengthening GC$^-$ to the sole requirement of $N_i(g) = N_i(g')$ and $g \subseteq g'$, as a seemingly natural counterpart to GC$^+$, does not provide a suitable monotonicity concept. It would, for instance, consider the removal of link $\{2, 3\}$ in 5-player simple majority voting on a left-right scale (section 3.4) as advantageous for player 1 while, in fact, it turns 1 into a null player of v/g.

[27]Hamiache (2011) uses a weakened version of GC$^-$, *central player monotonicity*, for a characterization of MV. It concerns link losses of players $j \neq i$ under the additional premise that i can communicate directly with all other players. Slikker (2005b) investigates the monotonicity requirement that additional links never lower the power of *any* player. This is stronger than GC$^+$, has a character almost opposite to GC$^-$, and is satisfied by none of the considered indices on $\mathcal{W}^{\mathcal{G}}$.

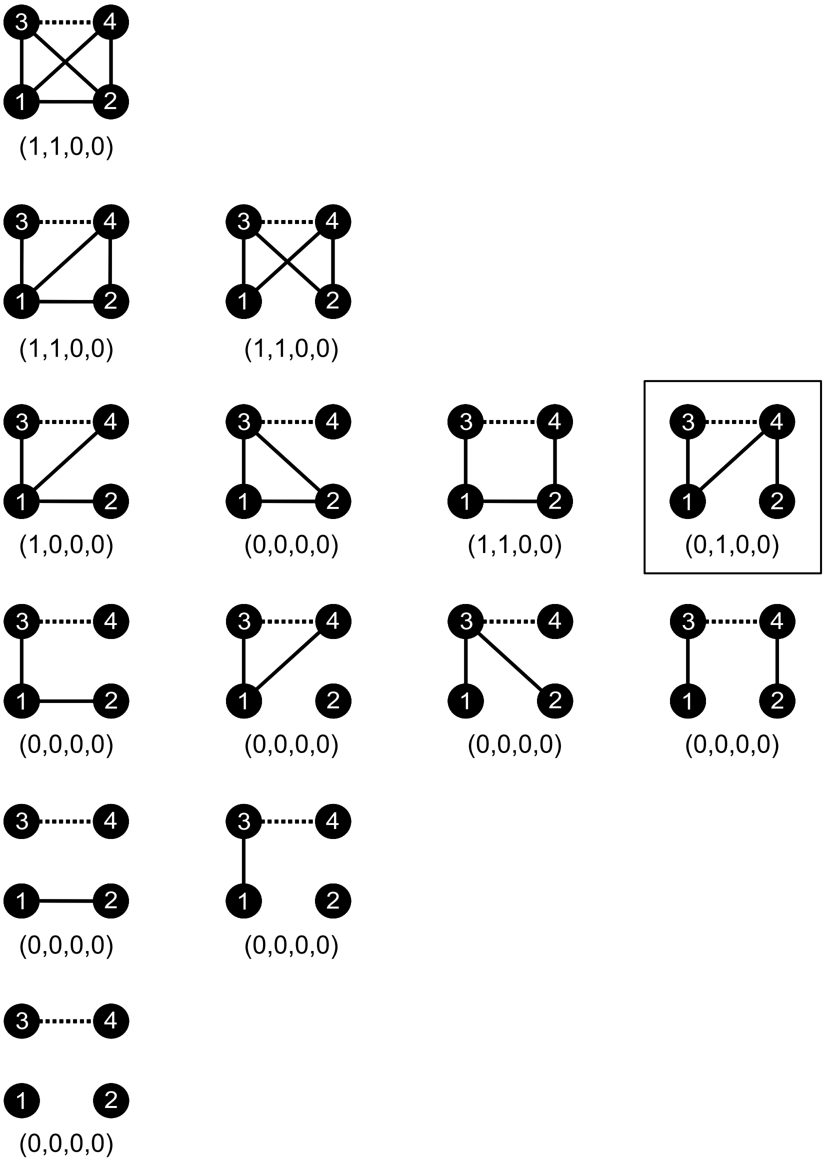

FIGURE 3. Definition of a symmetric power index which satisfies GC^+ and GC^- but not LC for $(N, v) = [2; 1, 1, 0, 0]$

for a particular assignment of players 1 and 2 to the bottom and players 3 and 4 to the top nodes, respectively; those for other communication structures follow by anonymity. Power of players 3 and 4 is defined to be zero irrespectively of the communication graph. This violates neither SYM nor GC^+ nor GC^-. The power assigned to players 1 and 2 varies with the communication structure (but is independent of whether the dashed link between 3 and 4 is included). The assignment respects symmetry. And, concerning inter-graph comparisons, it also respects GC^+ and GC^- for players 1 or 2. However, the fourth graph in the third row involves a violation of LC: player 1 has less rather than at least as much power as player 2.

The reason why a clear-cut relationship between local and global monotonicity exists with respect to weights but not for communication possibilities lies in the different nature of these two resources. Each unit of voting weight is perfectly divisible and a perfect substitute for any other one. This implies that two players who are to be compared can be given an equal weight by shifting some from the stronger to the weaker one. GW predicts that a transfer of voting weight from one player to another is weakly beneficial for the recipient and weakly detrimental for the donor. As they become symmetric, and have equal power by SYM, the desired local monotonicity follows. In contrast, communication structures are of discrete nature. In a situation where, for instance, two players are identical apart from one player having a link in advance (players 1 and 2 in the boxed graph of figure 3), they can only be rendered symmetric by deleting a link of the stronger player, or adding a link for the weaker one, which does *not* involve the respective other player. The latter is necessarily a "third party" in the equalization process. No notion of global monotonicity, which compares communication structures from the perspective of an individual player, can exclude or sign externalities on third parties. Such an externality precludes the deduction of a locally monotonic power ranking for the boxed graph of figure 3: although the defined power index satisfies GC^+ and GC^-, player 1 gains more power than 2 when link $\{2,3\}$ is added. Similarly, player 2 loses more than 1 when link $\{1,4\}$ is removed. Without a constraint that involves a comparison of gains from a link addition (or losses from a deletion) across players that are directly involved and those that are only indirectly involved, there is no way to deduce that player 1 must have at least as much power as player 2, even though they are symmetric, and hence have equal power, in the respective neighbors of the boxed graph.[28]

5.2. **Properties of the Considered Indices.** The restricted games $(N, v/g)$ and $(N, v/g')$ which are induced by two comparable communication structures g and g' preserve the notion of g offering better communication possibilities for a given player than g' which is underlying GC^+ or GC^-. Similarly, that some player has better communication possibilities than another one locally, i.e., in a given graph, translates into a restricted game which is better for one player than the other. Power indices for communication structures that are defined as a semivalue of the respective restricted game, therefore, satisfy all of the monotonicity notions defined above.

Proposition 4. *The restricted versions of all semivalues – so, in particular, the Myerson value MV and restricted Banzhaf index RBI – are LC-monotonic, GC^+-monotonic, and GC^--monotonic.*

Proof.
(LC) Consider (N, v, g) such that g offers better communication for i than for j in the sense of LC. Then there is a subgraph $g' \subseteq g$ with $g'|_{N\setminus\{i\}} = g|_{N\setminus\{i\}}$ such that i and j are symmetric in (N, v, g') with respect to permutation π with $\pi(i) = j$. For this it holds $v/g(S \cup \{i\}) \geq v/g(\pi(S) \cup$

[28]In a previous version of this paper (Napel et al. 2011), we study the requirement that, when a player gets additional communication possibilities, his gain must be at least as high as that of any other player. This property implies GC^+ in the presence of CE, and LC in the presence of SYM.

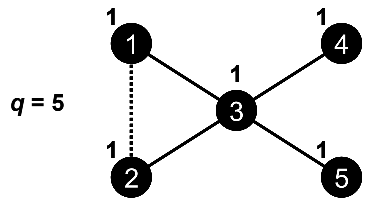

FIGURE 4. Position value violates LC, GC^+, and GC^- ($PV_1 \not\geq PV_4$ with $\{1,2\}$; player 1 gains and 4 loses when $\{1,2\}$ is removed)

$\{j\}$) and $v/g(S) \leq v/g(\pi(S))$ for all $S \not\ni i$. Hence, $v/g(S \cup \{i\}) - v/g(S) \geq v/g(\pi(S) \cup \{j\}) - v/g(\pi(S))$ for all $S \not\ni i$. This yields the desired inequality for the semi-values of i and j.

(GC^+) Consider $(N, v, g), (N, v, g') \in \mathcal{W}^{\mathcal{G}}$ such that g offers better communication possibilities for player i in the sense of GC^+. Then for any $S \not\ni i$, it holds $v/g(S \cup \{i\}) \geq v/g'(S \cup \{i\})$ and $v/g(S) = v/g'(S)$. Therefore, it is $v/g(S \cup \{i\}) - v/g(S) \geq v/g'(S \cup \{i\}) - v/g'(S)$ and the desired inequality follows directly from the definition of a semivalue (see equation (3)).

(GC^-) Consider $(N, v, g), (N, v, g') \in \mathcal{W}^{\mathcal{G}}$ such that g offers better communication possibilities for player i in the sense of GC^-. For any $S \not\ni i$, choose $T \in (S \cup \{i\})/g$ with $T \ni i$. For this we also have $T \in (S \cup \{i\})/g'$ and thus $v/g(T) = v/g'(T)$. Moreover, $v/g(T \backslash \{i\}) \leq v/g'(T \backslash \{i\})$. Thus, $v/g(S \cup \{i\}) - v/g(S) = v/g(T) - v/g(T \backslash \{i\}) \geq v/g'(T) - v/g'(T \backslash \{i\}) = v/g'(S \cup \{i\}) - v/g'(S)$. The desired conclusion follows. □

The position value does not satisfy *any* of the communication monotonicities, not even in perfectly symmetric voting situations. To see this, consider five player unanimity voting $(N, v) = [5; 1, 1, 1, 1, 1]$ with graphs $g = \{\{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{3,5\}\}$ and $g' = g \backslash \{\{1,2\}\}$ (see figure 4). With graph g, players 1 and 2 share player 4's and 5's sole possibility to communicate with player 3 and should hence have at least as much power according to LC. However, 4's and 5's single links function as veto players in the link game while 1's communication possibilities (and 2's) in some sense "compete" with each other – the position value amounts to $PV(N, v, g) = (0.1, 0.1, 0.45, 0.175, 0.175)$ and violates LC.[29] In the transition from g to g', players 1 and 2 lose the communication possibility between them, which should be detrimental for these players according to GC^+. Also, according to GC^-, it should be weakly beneficial for players 4 and 5 since no cooperation possibility is lost for them. However, the position value evaluates to $PV(N, v, g') = (0.125, 0.125, 0.5, 0.125, 0.125)$, in violation of GC^+ and GC^-.

The average tree solution, as defined by Herings et al. (2010), does not satisfy any of the communication monotonicities either. Consider 4-player unanimity voting $(N, v) = [4; 1, 1, 1, 1]$ with graphs $g = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}\}$ and $g' = g \backslash \{\{2,3\}\}$ (see figure 5). With unanimity voting, a player's power is proportional to the number of admissible spanning trees rooted in it. For communication structure g, there are four admissible spanning trees

[29]That violations of monotonicity arise due to "competition" between several links of one agent suggests to consider some kind of agent-based a priori unions in the link game. This, in fact, leads to MV (see fn. 16).

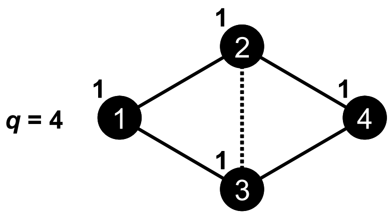

FIGURE 5. Average tree solution violates LC, GC^+, and GC^- ($ATS_2 \not\gtrsim ATS_4$ with $\{2,3\}$; player 2 gains while 4 loses when $\{2,3\}$ is removed)

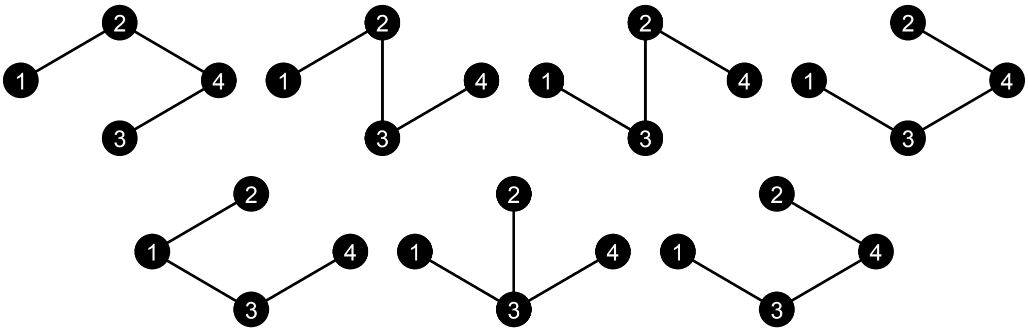

FIGURE 6. Admissible spanning trees rooted in players 1 (top) and 2 (bottom) for the communication structure in figure 5 with $\{2,3\}$

with players 1 or 4 as the root while there are three rooted in players 2 or 3 (see figure 6). The average tree solution, therefore, amounts to $ATS(N,v,g) = (\frac{4}{14}, \frac{3}{14}, \frac{3}{14}, \frac{4}{14})$ – in violation of LC which requires players 2 and 3 not to be worse off than 1 and 4. Removal of link $\{2,3\}$ makes all players symmetric such that $ATS(N,v,g') = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. This violates GC^+, according to which the loss of their communication possibility should not be beneficial for 2's and 3's power. It also violates GC^-, which requires that 1 and 4 are not harmed by a loss of communication possibilities between players that they can communicate with directly.

6. DISCUSSION

Power indices for voting games with restricted communication condense two-dimensional resources – weights and communication possibilities – into single numbers. This is bound to involve implicit trade-offs across the weight and communication dimensions, which blur intuitions about how power indications should change with, for instance, the removal of a link. Well-defined notions of monotonicity can help. They (i) clarify vague intuitions, (ii) allow to transparently connect distinct requirements for how power indices should behave *ceteris paribus*, and (iii) classify indices according to which "better than"-relations they preserve.

Because power indices for communication structures evaluate both weight and communication resources, they can, trivially, also be used for the analysis of *only weight* differences, or of

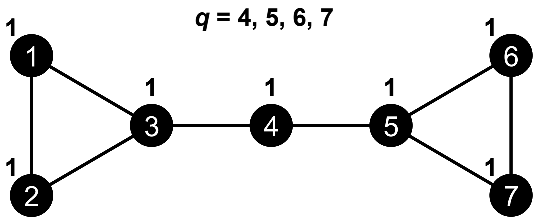

Figure 7. Communication structure from Jackson (2008, p. 38) with different majority rules

$q = 4$:

i	MV_i	RBI_i	PV_i	ATS_i
1, 2, 6, 7	0.06	0.08	0.08	0.00
3, 5	0.17	0.23	0.22	0.00
4	0.42	0.36	0.27	1.00

$q = 5$:

i	MV_i	RBI_i	PV_i	ATS_i
1, 2, 6, 7	0.03	0.05	0.02	0.00
3, 5	0.30	0.17	0.24	0.43
4	0.30	0.17	0.43	0.14

$q = 6$:

i	MV_i	RBI_i	PV_i	ATS_i
1, 2, 6, 7	0.07	0.05	0.04	0.07
3, 5	0.24	0.08	0.21	0.29
4	0.24	0.08	0.35	0.14

$q = 7$:

i	MV_i	RBI_i	PV_i	ATS_i
1, 2, 6, 7	0.14	0.02	0.07	0.14
3, 5	0.14	0.02	0.18	0.14
4	0.14	0.02	0.29	0.14

Table 2. Power in voting games with communication structures described in figure 7

only communication differences. With respect to weights, restricted versions of standard power indices, such as the Shapley-Shubik index, simply return a known index when they are applied to the full graph; others define "new" power indices for unrestricted voting games. Analogously, following an idea of Owen (1986), one might measure the advantageousness of particular network locations, often identified with the *centrality* of players in a communication graph, by applying a power index for communication structures to a symmetric voting situation, i.e., a voting rule such as simple majority or unanimity voting.

Table 6 reports the Myerson value, restricted Banzhaf index, position value and the average tree solution for the communication structure shown in figure 7 (adapted from Jackson 2008, p. 38) and, respectively, simple majority voting ($q = 4$), different supermajority rules ($q = 5$ and $q = 6$), and unanimity rule ($q = 7$). It can be seen that the ranking of players depends on *which* symmetric voting rule is considered: ties are created or broken in case of MV and RBI; a strict ordering is reversed for ATS. This should not be too surprising because the strategic environment of the agents changes significantly with the majority threshold.[30] But it is worthwhile to keep this observation in mind.

[30]This can, e.g., be seen by considering the *core* of the corresponding restricted game. For simple majority ($q = 4$), the core of $(N, v/g)$ equals $\{(0, 0, 0, 1, 0, 0, 0)\}$. For $q = 5$ or 6, it contains all imputations which assign zero to players 1,2, 6, and 7. And for unanimity ($q = 7$), every imputation is in the core.

	MV	*RBI*	*PV*	*ATS*
SYM	+	+	+	+
CE	+	−	+	+
LW	+	+	−	+
GW	+	+	−	+
LC	+	+	−	−
GC^+	+	+	−	−
GC^-	+	+	−	−

Table 3. Properties of the considered indices

The most prominent amongst the many measures of centrality in social networks – *degree centrality*, *Katz prestige*, and *Bonacich* or *eigenvector centrality* – rank players 3 and 5 as the most central in figure 7, followed by player 4, and finally players 1, 2, 6, and 7 (Jackson 2008, p. 43). The only power indications which support this ranking are provided by *ATS* for $q = 5$ and $q = 6$.[31] This indicates one reason why we did not follow a tempting route to the possible formalization of monotonicity with respect to communication possibilities: namely, to require a player's power to be weakly increasing *ceteris paribus* in a given measure of centrality. None of the established indices would satisfy this, at least for the most straightforward centrality measures. What speaks more generally against the development of monotonicity notions based on some measure of centrality is that these measures themselves involve a considerable dimensionality reduction and need to be checked against various intuitive notions of when one player should be regarded more central than another. Still, future research might make useful attempts in this direction – and perhaps comes up with new interesting values by investigating extensions of centrality indicators.

The properties of the considered indices are summarized in table 3. We find that the Myerson value and the restricted Banzhaf index are most in line with the proposed notions of monotonicity. They satisfy all requirements that we have defined here; and when they violate yet more demanding ones, so do their peers. That the position value violates *all* of the monotonicities considered in this paper should call for some caution – especially because any violation of communication monotonicities *a fortiori* pertains to the entire class of superadditive TU games with restricted communication. By always dividing the power of a link half-half and, more importantly, by allowing for a detrimental kind of competition between the links of a single agent, *PV* can produce puzzling rankings. Herings et al.'s average tree solution satisfies both weight monotonicities but none of the three considered communication monotonicities. Identification of *ATS*'s non-monotonicity and of the underlying reasons (primarily, the restriction to admissible rooted spanning trees) helps to motivate alternative average tree solutions. We show in the appendix that

[31]The coincidence has to do with the graph's linear middle part. For linear communication (sub)structures, *ATS* favors players that are pivotal from either end of the graph. This benefits the median, player 4, under simple majority, and more peripheral ones for $q = 5$ or 6.

a variation of Herings et al.'s definition, which is based on *all* possible rooted spanning trees, satisfies two of the suggested requirements at least in situations in which no cycles are directly involved in player or game comparisons.

Appendix. The Unrestricted Average Tree Solution

In the main text, we examined the average tree solution proposed by Herings et al. (2010), which restricts the averaging to a particular class of admissible rooted spanning trees. Baron et al. (2011), however, have introduced and axiomatized average tree solutions for all possible classes of rooted spanning trees. In this appendix, we consider the monotonicity properties of the "canonical" *unrestricted* variant which is based on the class of *all* possible rooted spanning trees.

Define the *unrestricted average tree solution* ATS^* by

$$ATS^*_i(N, v, g) \equiv \frac{1}{|T^*_{S_i,g}|} \sum_{(t,j)\in T^*_{S_i,g}} m^v_i(t, j), \quad i = 1, \ldots, n, \tag{11}$$

where $S_i \in N/g$ is the component containing i and $T^*_{S_i,g}$ is the set of *all* rooted spanning trees on S_i for g. ATS^* satisfies component efficiency and symmetry. It is also LW and GW-monotonic (see fn. 24). In case of a unanimity game (N, v) with any graph g that connects the grand coalition, ATS^* assigns equal power to each player (in contrast to ATS, which does not do so and even violates the examined communication monotonicities for unanimity games). It indicates positive power for null players if (N, v) has no dictator.

The unrestricted average tree solution satisfies LC at least when the communication possibilities that a player i has in advance of j connect him only to players which would without the links that i has in advance belong to a different component. Similarly, it satisfies GC^+ at least when the additional neighbors of a player i in game g are not a member of his component in g'. Before we formalize this, note that the average tree solution of Herings et al. does *not* satisfy LC or GC^+ in these situations: consider $(N, v) = [2; 1, 1, 1, 0]$, i.e., simple majority voting amongst players 1, 2 and 3, with player 4 added as a null player. With a full graph $g = g^{\{1,2,3\}}$ on the first three players, one obtains $ATS(N, v, g) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$. Then, with link $\{1, 4\}$ added, i.e., considering $g' = g \cup \{\{1, 4\}\}$, GC^+ calls for player 1's power to be weakly greater with g' than with g. Moreover, LC requires 1's power to be no less than 2's or 3's with communication structure g'. But $ATS(N, v, g') = (\frac{2}{8}, \frac{3}{8}, \frac{3}{8}, 0)$ in violation of GC^+, and LC. In contrast, ATS^* does not violate LC or GC^+ here: independent of link $\{1, 4\}$, it is $ATS^*(N, v, g) = ATS^*(N, v, g') = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$.

Proposition **5.** *Denote the component of N containing i in graphs g and g′ by S_i and S'_i, respectively. (i) For each $(N, v, g) \in \mathcal{W}^{\mathcal{G}}$,*

$$ATS^*_i(N, v, g) \geq ATS^*_j(N, v, g)$$

holds for all players i and j for which there is a graph g' verifying the qualifying condition in LC, i.e.,

(a) $g' \subseteq g$ and $g'|_{N\backslash\{i\}} = g|_{N\backslash\{i\}}$,

(b) i and j are symmetric in (N, v, g') ,

and, in addition, satisfies

(c) $\{k, k'\} \notin g$ for all $k \in S_i \backslash S'_i$ and $k' \in S'_i \backslash \{i\}$.

(ii) For all $(N, v, g), (N, v, g') \in \mathcal{W}^{\mathcal{G}}$,

$$ATS^*_i(N, v, g) \geq ATS^*_i(N, v, g')$$

holds whenever the qualifying condition in GC^+ is verified, i.e.,

(a) $g \supseteq g'$ and $g|_{N\backslash\{i\}} = g'|_{N\backslash\{i\}}$,

and, in addition,

(b) $\{k, k'\} \notin g'$ for all $k \in S_i \backslash S'_i$ and $k' \in S'_i \backslash \{i\}$.

Proof.

(i) First, if $S_i = S'_i$, then $ATS^*_i(N, v, g) = ATS^*_j(N, v, g)$ by symmetry. So assume $S_i \supset S'_i$. Let π with $\pi(i) = j$ be a permutation such that i and j are symmetric in (N, v, g'). Without loss of generality, let $\pi(k) = k$ for $k \notin S'_i$. Every spanning tree t on S_i can be uniquely partitioned into disjoint spanning trees t' on S'_i and t'' on $S_i \backslash S'_i \cup \{i\}$. And, conversely, the union of disjoint spanning trees t' on S'_i and t'' on $S_i \backslash S'_i \cup \{i\}$ yields a spanning tree on S_i. Due to symmetry on S'_i, also $\pi(t') \cup t''$ is a spanning tree on S_i and the transition from $(t' \cup t'', k)$ to $(\pi(t') \cup t'', \pi(k))$ constitutes a 1-to-1 mapping on $T^*_{S_i,g}$. Now the inequality

$$m^v_i(t' \cup t'', k) \geq m^v_i(t', i) = m^v_j(\pi(t'), j) \geq m^v_j(\pi(t'), i) = m^v_j(\pi(t') \cup t'', k)$$

holds for any spanning tree $(t' \cup t'', k)$ on S_i and $k \in S_i \backslash S'_i$. And for $k \in S'_i$,

$$m^v_i(t' \cup t'', k) = m^v_j(\pi(t') \cup \pi(t''), \pi(k)) \geq m^v_j(\pi(t') \cup t'', \pi(k)).$$

It follows that $ATS^*_i(N, v, g) \geq ATS^*_j(N, v, g)$.

(ii) First, if $S_i = S'_i$, then $g = g'$ and, trivially, $ATS^*_i(N, v, g) = ATS^*_i(N, v, g')$. So assume $S_i \supset S'_i$. Every spanning tree t on S_i can be uniquely partitioned into disjoint spanning trees t' on S'_i and t'' on $S_i \backslash S'_i \cup \{i\}$. And, conversely, the union of any disjoint spanning trees t' on S'_i and t'' on $S_i \backslash S'_i \cup \{i\}$ yields a spanning tree on S_i. Consider a given spanning tree $t = t' \cup t''$ on S_i and note, first, that for any $k \in S'_i$

$$m^v_i(t' \cup t'', k) \geq m^v_i(t', k).$$

Second, for $k \in S_i \backslash S'_i$ we have

$$m^v_i(t' \cup t'', k) \geq m^v_i(t', i) = \max_{k \in S'_i} m^v_i(t', k).$$

It thus follows that the average marginal contribution for spanning tree $t = t' \cup t''$, taken over all roots $k \in S_i$, is weakly greater than the average marginal contribution for spanning tree t', taken over all roots $k \in S'_i$. Since every spanning tree t' on S'_i corresponds with a constant number $|T^*_{S_i \setminus S'_i \cup \{i\}, g}|$ of spanning trees $t = t' \cup t''$ on S_i, it follows that $ATS^*_i(N, v, g) \geq ATS^*_i(N, v, g)$. □

In view of the combinatorial difficulties involved with spanning trees in the presence of cycles, we do not have a conjecture regarding the satisfaction of LC or GC^+ by ATS^* for arbitrary communication situations or arbitrary changes in communication possibilities.

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