The Power of a Spatially Inferior Player*

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Abstract

Traditional power indices are not suited to take account of explicit preferences, strategic interaction, and particular decision procedures. This paper studies a new way to measure decision power, based on fully specified spatial preferences and strategic interaction in an explicit voting game with agenda setting. We extend the notion of inferior players to this context, and introduce a power index which – like the traditional ones – defines power as the ability to have pivotal influence on outcomes, not as the (often just lucky) occurrence of outcomes close to a player's ideal policy. Though, at the present state, formal analysis is based on restrictive assumptions, our general approach opens an avenue for a new type of power measurement.

1 *Introduction*

Power is an important concept in the analysis of economic and political institutions, and even of moral codes and ethics. Though everybody has some understanding about who under what circumstances exerts power, the

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concept is elusive. Therefore, it is not surprising that there is considerable controversy as to what constitutes an appropriate measure of power even in the restricted class of those economic or political institutions which can be represented as simple games in coalitional form.

Power indices assign to each player of a n-person simple game, such as a weighted multi-party voting game, a non-negative real number which indicates the player's a priori power to shape events. Numerous indices have been proposed – most notably by Shapley and Shubik (1954), Banzhaf (1965), Deegan and Packel (1978), and Holler and Packel (1983).[1] On the surface, the distinction between these is whether minimal winning coalitions, crucial coalitions, player permutations, or other concepts are the primitives for measurement. More fundamentally, the discussion is about the realism of the distinct probability models behind alternative indices, desired properties like monotonicity and, importantly, the congruence of indications for basic reference cases with predictions by other tools of economic or political analysis.

In this light, the following basic example is striking. Consider the 3-player simple game where the only winning coalitions are the grand coalition ABC and the two coalitions AB and AC. A could be the federal government that needs approval from one of two provincial governments to pass laws. Or, A might be a shareholder who needs to be backed by at least one of two (smaller) shareholders to decide on strategic questions of corporate policy. Economic equilibrium analysis would claim A to be "on the short side of the market", implying that B and C cannot influence terms of trade. From the point of non-cooperative game theory, A can be imagined to make an ultimatum offer to B, asking for approval in return for an only marginal (and in the limit non-existent) concession to B's political or economic interests. A rational player B would have to accept since a potential threat of colluding with C to obtain a better deal is not credible or subgame perfect. A symmetric argument applies to C. Drawing on cooperative game theory, the core and nucleolus of this game are both $\{(1, 0, 0)\}$ and further support the intuition that B and C are powerless in this game. Despite this clear prediction by different types of non-cooperative and cooperative game-theoretic reasoning, the power indices of Banzhaf and Shapley-Shubik indicate substantial power for powerless players B and C. They yield the power vectors $(\frac{3}{4}, \frac{1}{4}, \frac{1}{4})$ and $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$, respectively. In Napel and Widgrén (2001a) the notion of *inferior*

[1]For recent comparative investigations of power indices, their properties and applicability, see Felsenthal and Machover (1998) and Holler and Owen (2001).

players was defined to reach a more satisfactory solution.

To put it in a more general context, the criticism that power indices usually face stems from two factors. First, closely related to our example above, traditional power indices do not take players' strategic interaction into account. Second, their capability of modelling complicated institutional features, like agenda-setting, is limited. The inferior player axiom and the strict power index derived from it are an attempt to tackle these problems. In this paper, we take one further step and attempt to define the concept of inferior players in a spatial voting context. Our goal is then to build an a priori measure of power, which corresponds with the strict power index and which opens the avenue for taking preferences into the analysis of power. Moreover, the approach allows us to model more complex institutional features of the game, like agenda setting.

The fundamental difference between spatial voting and coalitional form games is that the latter has the set of players and the former the set of policy outcomes as the domain. In spatial voting, players are supposed to have ideal points in a policy space and payoff is assumed to be monotonically decreasing in the distance between ideal policy outcome and actual one. In coalitional form games, coalitions rather than individuals gain when a coalition is able to pass proposals. Power indices then give estimates for an individual's influence on a coalition's achievement. In this paper, we discuss this difference and aim to take both approaches into account.

Recently, strategic aspects and power indices have been studied by Steunenberg et al. (1999). In their analysis, the *strategic power index (StPI)*[2] of player i, Ψ_i, is defined as

$$\Psi_i = \frac{\Delta_d - \Delta_i}{\Delta_d}$$

where Δ_d is the expected distance between the equilibrium outcome and the ideal point of a dummy player, and Δ_i is the expected distance between the equilibrium outcome and the ideal point of player i. In Steunenberg et al. (1999), a dummy is a player who is like an outside observer of the game having no power. The ideal policy of a dummy player is assumed to vary within the same range as the ideal points of the actual players of the game but a dummy does not have any decision making rights and, thus, she does

[2] We use this abbreviation instead of SPI to avoid confusion with the *Strict Power Index*, which is defined in Napel and Widgrén (2001a) and abbreviated as SPI.

not matter for the outcome of the game.[3] From the formula it is easy to see that a player who always gets her ideal policy obtains one as the power value and a player who is like a dummy gets zero as her power value.

In the case of the strategic power index, the introduction of a dummy player is due only to standardisation. The distance Δ_i plays the key-role. Without normalisation, Δ_i would take the role of an "absolute" power measure. The underlying idea is simply to define power on the basis of proximity between players' most preferred positions and actual outcomes. At first glance this may sound appealing but there is at least one caveat. Given that players' preferences are spatial a voter may well have an ideal point very close to the outcome although the passage of the proposal that has lead to this outcome was completely out of her control. Proximity is often due to luck, not power. Let us illustrate things with a simple example.

Consider a seven-player symmetric perfect information voting game, with player set $\{A, B, C, D, E, F, G\}$ and a 5/7th majority rule. Assume ideal points in a uni-dimensional policy space which order the players' positions from left to right as follows: $ABCDEFG$. Consider a proposal χ which is located in between E's and F's ideal points, but closer to E than F. StPI suggests that if χ is accepted, then E exerts more power in this preference configuration than players A, B, C, D, F and and G. However, the outcome of this vote depends on the location of the current state of affairs, i.e. status quo. For simplicity, suppose that status quo lies left of A. Coalition $ABCDE$ is then a potential minimal winning coalition, as well as $BCDEF$ and $CDEFG$. Consider the first alternative. Given the locational assumptions $ABCDE$ cannot be minimal winning with respect to proposal χ in a spatial sense since if the players in it accept proposal χ, then so do F and G. Player A is the most likely member of $ABCDE$ to reject the proposal χ but is no longer critical given F's and G's acceptance. We get $BCDEF$ as the next candidate minimal winning coalition. The same argument as before holds for this coalition – it is not a minimal winning coalition in a spatial sense since if it were approving proposal χ then G would also accept it. Consider $CDEFG$. In this coalition, E is closer to the proposal than any other player. But is this due to her power? The only player who has a credible swing in this coalition is C. Only if status quo is further to the left from C than χ is to the right, C accepts the proposal. Player E does not have such position for this preference configuration. In fact, this holds for nearly all

[3]Note that this is not the standard way to define a dummy player.

proposals and locations of status quo.

In this paper, we take an alternative approach to strategic power and follow internal rather than external normalisation. This means that whether a player is dummy or not depends on her capabilities in the game. Contrary to Steunenberg et al. (1999), we assume that any player, dummy or not, is an actual player and not an external observer. We define *a posteriori power* as having an effective pivotal position for a *given* preference configuration, and *(a priori) power* as the ex ante expectation of it, taken with respect to the probabilities of different preference configurations. This allows for different informational considerations and makes the analysis more procedural than in the case of StPI. Our approach leads to a definition of power, which, in fact, corresponds to that of established power indices.

2 *Coalitional Form and Spatial Voting Games*

Coalitional form voting games deal with all possible coalitions of members of a set $N \equiv \{1, \ldots, n\}$ of players. Players' preferences are not known. Coalitions $S \subseteq N$ are either winning or losing, implying a partition of the set of all coalitions, $\mathcal{P}(N)$, into the set $\mathcal{W}$ of *winning coalitions* and the set $\mathcal{L}$ of *losing coalitions*.[4] A *coalitional form voting game* is a special instance of a *simple game* defined by the pair $(N, \mathcal{W})$, where the set of winning coalitions, $\mathcal{W}$, can be characterized by a non-negative real vector $r_v = (m; w_1, \ldots, w_n)$, where w_i is player i's number of votes and m is the number of votes that establishes a winning coalition. In a *simple majority voting game*, $w_i = 1$ for every player $i \in N$ and $m = n/2 + 1$ or $m = (n + 1)/2$ for even or odd n, respectively.

A game $(N, \mathcal{W})$ can equivalently be described by its *characteristic function* v. It maps n-tuples $s \in \{0, 1\}^n$, which represent a feasible coalition $S \subseteq N$ by indicating which players $i \in N$ belong to S $(s_i = 1)$ and which do not $(s_i = 0)$, either to $v(S) = 1$ if $S \in \mathcal{W}$ or to $v(S) = 0$ if $S \in \mathcal{L}$. When $\mathcal{W}$ represents winning coalitions in a voting game, v is monotonic, i. e. $v(S) = 1$ implies $v(T) = 1$ for any superset $T \supseteq S$.

[4]We only consider *proper games* in which the complement of a winning coalition is losing, i. e. $S \in \mathcal{W} \Rightarrow N - S \in \mathcal{L}$. We do not assume that the game is *decisive*, i.e. additionally $N - S \in \mathcal{L} \Rightarrow S \in \mathcal{W}$, because this would preclude the analysis of qualified majority voting. If both $S \in \mathcal{L}$ and $N - S \in \mathcal{L}$, then status quo prevails (see definition below).

A player who by leaving a winning coalition $S \in \mathcal{W}$ turns it into a losing coalition $S - \{i\} \in \mathcal{L}$ has a *swing* in S. He is called a *crucial* or *critical member* of coalition S. Coalitions in which at least one member is crucial are called crucial coalitions.[5] Coalitions where player i is critical are called *crucial coalitions with respect to* i. Let

$$C_i(v) \equiv \{S \subseteq N \mid v(S) = 1 \wedge v(S - \{i\}) = 0\}$$

denote the set of crucial coalitions w.r.t. i. The number of swings of player i in simple game v is thus

$$\eta_i(v) \equiv \mid C_i(v) \mid .$$

A player i who is never crucial, i. e. $\eta_i(v) = 0$, is called *dummy player*. In Napel and Widgrén (2001a), the following related concept is introduced:

Definition 1 *Player i is* inferior *in simple game v if* $\exists j \neq i$:

$$\begin{array}{rl} \forall S \in C_i(v): & j \in S \\ \wedge \;\; \exists S' \in C_j(v): & i \notin S' \end{array}$$

An inferior player i is equivalently characterized by $C_i(v) \subsetneq C_j(v)$ for $j \neq i$.[6] It is straightforward to see that every dummy player is inferior but the reverse does not hold (see Napel and Widgrén 2001a). Let us refer to a player who is not inferior as *superior*.

The game with $\mathcal{W} = \{AB, AC, ABC\}$ was used above to illustrate the divergence between power predictions based on conventional indices, on the one hand, and competitive analysis or the concept of the core of a game, on the other hand. Imagine that the spoils of a winning coalition in v are \$100 and to be split among its members. Alternatively, consider 100 policy units, e. g. referring to different topics in a policy proposal for each of which the

[5]Deegan and Packel (1978) use the term 'minimal winning coalition', Felsenthal and Machover (1998) the term 'vulnerable coalition' instead of 'crucial coalition'. We, like other authors, follow Bolger's (1980) conceptualization.

[6]Note that the first part of Definition 1 implies that j belongs to all *minimal* winning coalitions to which i belongs and is (by definition) crucial in these. Assuming that i has a swing in a non-minimal winning coalition without j also having one leads to a contradiction once non-crucial members of that coalition – including j – are dropped. Therefore, i does never have a swing without j also having a swing in the same coalition – but j has at least one swing in a coalition without i having one.

players have distinct preferred alternatives. Regardless of the precise object of conflicting interests in v, player A is in the position of the proposer in a non-cooperative *Ultimatum Game* with B as responder when the situation permits negotiations before the final establishment of a winning coalition. Since A has the option to form a winning coalition without B, B cannot do better but to accept whatever A proposes in terms of B's share of spoils or political influence. A anticipates this and rationally offers B a share of (almost) nothing. The Banzhaf index of this game is $\left(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}\right)$. In Napel and Widgrén (2001a) this was corrected by replacing the conventional dummy player axiom of power measurement with a corresponding inferior player axiom. In the example, we get the following strict power index $\left(\frac{3}{4}, 0, 0\right)$.

It is worth noting that in the spatial context considered in this paper things are different. Despite the fact that the agenda-setter is superior to all voters, the game is *not* necessarily a pure ultimatum game. Basically, this is due to the possible veto power exerted by the voters. The equilibrium outcome of the game depends on the pivotal player's preferred point. This implies that a pivot may be able to put credible threats on the agenda-setter, despite being inferior in a coalitional form, non-spatial sense.

In coalitional form games players' preferences do not have any role in determining the outcome. A usual way to justify this is to say that coalitional form voting games analyse institutions rather than actual votes and that there is no sufficient a priori information about players' preferences. Games in coalitional form thus analyse several votes.

Coalitional form games usually do not model agenda setting either. To add agenda setting into our model let us distinguish between two types of agents, namely fixed agenda setters $j \in A$ and voters $i \in N$. In spatial voting games, players' preferences restrict the class of feasible winning coalitions.

Definition 2 *A* one-dimensional $(n+s)$-player spatial voting game with agenda setting *is a 5-tuple* $(N, A, \mathcal{W}, \Lambda, \sigma)$ *where* N *is the set of voters,* A *is the set of agenda setters,* $\mathcal{W}$ *describes the class of majority coalitions needed for the passage of agenda setters' proposals,* $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ *is the vector of voters' ideal points, and* $\sigma \in \mathbb{R}^s$ *is the vector of agenda setters' ideal points.*

Throughout this paper, we assume that A is a singleton, hence $s = 1$. In general, however, it is easy to find examples where the agenda is set by a group of agents using pre-determined rules how to decide upon the agenda. The European Commission serves as an example.

We simplify the model by restricting the analysis to only one policy dimension. We also disregard weighted voting for the sake of simplicity. Given two policies x and y, ideal points partition N into

$$\begin{aligned} N_{x \succsim y} &\equiv \{i \,|\, d(x,\lambda_i) \leq d(y,\lambda_i)\} \\ N_{x \prec y} &\equiv \{i \,|\, d(x,\lambda_i) > d(y,\lambda_i)\} \end{aligned}$$

where $d(a,b) \equiv \sqrt{a^2+b^2}$ denotes the Euclidian distance between a and b. We normalize the *status quo* to $Q = 0$. Spatial preferences are e. g. represented by the utility functions $\pi_\sigma(\Omega) = -(\sigma-\Omega)^2$ and $\pi_{\lambda_i}(\Omega) = -(\lambda_i-\Omega)^2$, where Ω denotes the policy outcome of the game.

We think of voters' and the agenda setter's ideal points as being *ex ante* – when institutional a priori power is evaluated – random variables denoted by $\tilde{\Lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$ and $\tilde{\sigma}$. Their distributions are $F_{\tilde{\Lambda}}$ and $F_{\tilde{\sigma}}$, respectively. However, actual decisions and the pivotal positions that are our indicators of a posteriori power[7] are determined under complete information, i. e. for particular commonly observed realizations Λ and σ of ideal points. For any given preference configuration we consider the following *agenda setting game (ASG)*:

1. Agenda setter A makes a take-it-or-leave-it proposal $\chi = \chi(\Lambda, \sigma, m)$, where m is the number of voters whose acceptance is needed to pass a proposal.

2. Voters $i \in N$ simultaneously accept or reject the proposal. The outcome of the game is $\Omega = \chi$ if the proposal is accepted and $\Omega = 0$ if the proposal is rejected.

A possible policy space of this game is shown in Figure 1. Suppose that voters' ideal points λ_i are a priori uniformly distributed on $[\alpha, \beta]$ where $\alpha \leq 0$, $\beta > 0$. Agenda setter's ideal point σ is supposed to lie in $[0, \beta]$ and it is assumed to have uniform distribution. At first glance, this particular assumption may sound restrictive but, in fact, it is with little loss of generality. Assuming $\alpha = 0$ we get the special case where there is no asymmetry with respect to possible ideal points between voters and the agenda setter. Our desire is to generalise the assumption of identical domains for all players'

[7] We do not explicitly analyse agenda-setting power here. For a corresponding extension see the general framework in Napel and Widgrén (2002).

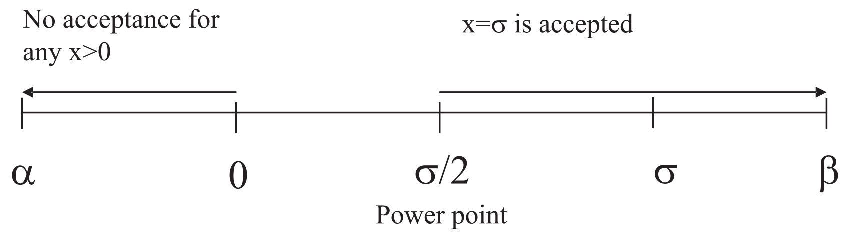

Figure 1: A simple uni-dimensional policy space

ideal points in a tractable way. For the simple procedural setting above it is natural to concentrate on asymmetry between voters and the agenda setter.

This allows for two kinds of interesting considerations. First, the interval $[\alpha, 0]$ gives the range where a voter does not gain from any proposal made by a rational agenda setter. It is a well-known result from spatial voting that players located in opposite directions from status quo do not cooperate. If we interpret the agenda setter as a seller and voters as buyers, then the interval $[\alpha, 0]$ gives the range where there are no gains from exchange; in a political context, the players have interests so conflicting that no mutually beneficial compromise about changing the status quo is possible. This can also be seen from the individual rationality constraints for acceptance that can be written

$$(\lambda_i - \chi)^2 \leq (\lambda_i - 0)^2 \tag{IR_i}$$

for voters $i \in N$ and correspondingly

$$(\sigma - \chi)^2 \leq (\sigma - 0)^2 \tag{IR_σ}$$

for the agenda setter A.

Second, asymmetrically distributed ideal points imply that the status quo bias plays a role in the model. This in turn makes it possible to investigate the effects of inefficiencies on power. Inefficiency emerges when a group of players is able to bloc any proposal made by the agenda setter whose IR_σ-constraint restricts the domain of proposals to $[0, \beta]$. This reduces power of both the agenda setter and those voters who would have preferred to replace status quo by some $\chi > 0$.

Using the assumptions above we get the following cumulative distributions functions

$$F_{\tilde{\lambda}_i}(x) = \begin{cases} 0, & \text{if } x \leq \alpha \\ \frac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha < x \leq \beta \\ 1, & \text{if } x > \beta \end{cases}$$

and

$$F_{\tilde{\sigma}}(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \frac{x}{\beta}, & \text{if } 0 < x \leq \beta \\ 1, & \text{if } x > \beta. \end{cases}$$

Note that

$$\hat{\lambda}_i \equiv \frac{\tilde{\lambda}_i - \alpha}{\beta - \alpha} \sim U(0,1).$$

For future use let us define the following re-scaling

$$\Pi(\sigma) \equiv \frac{\frac{1}{2}\sigma - \alpha}{\beta - \alpha}$$

and refer to it as the *power point.* The power point turns out to be the dividing line between cases when a player may exert power and when she may not. The range between the status quo and the power point is crucial for our concept of spatial inferiority. Note that a priori the power point is random. We get

$$\hat{\Pi} \equiv \frac{\frac{1}{2}\tilde{\sigma} - \alpha}{\beta - \alpha} \sim U\left(\frac{-\alpha}{\beta - \alpha}, \frac{\frac{1}{2}\beta - \alpha}{\beta - \alpha}\right).$$

Thinking of the players as representatives for some constituency or organization, it is reasonable to assume in the following that player i votes for a proposal χ whenever (IR_i) is satisfied. This means that after χ is proposed the coalition $N_{\chi \succsim 0} \subseteq N$ will form.

This assumption imposes considerable structure on the coalitions that are formed. Let (i) denote the player j whose ideal point, λ_j, turns out to be the i-th smallest of all voters so that $\lambda_{(1)} \leq \ldots \leq \lambda_{(n)}$. The agenda setter's rationality implies $\chi \geq 0$. Thus whenever $(IR_{(k)})$ is satisfied, then so are $(IR_{(k+1)}), \ldots, (IR_{(n)})$. Hence, any coalition which is formed is *convex* or *connected* in the following sense:

$$(i) \in S \wedge (i+l) \in S \;\Rightarrow\; (i+1), \ldots, (i+l-1) \in S, \qquad l \geq 2.$$

We will refer to a coalition with this property given a realized vector of voter's ideal points, Λ, as a Λ-*connected coalition.*[8]

[8]In our setting, one might use the more specific term Λ-*right*-connected coalition to stress that a formed coalition necessarily includes all players to the right of a given member.

Whether a particular Λ-connected coalition $S \subseteq N$ will be formed or not depends on the agenda setter's proposal χ. For given χ and Λ, there is a unique (χ, Λ)-*individually rational* or (χ, Λ)-*IR* connected coalition $S = N_{\chi \succsim 0}$ which will form. This may be winning or losing.

In a winning (χ, Λ)-IR coalition S, some players can have a swing in the traditional coalitional sense, i. e. can turn S into a losing coalition by leaving. In a spatial context, threatening to reject χ is generally no credible option e. g. for player (n), who is in fact the most eager to replace the status quo by χ. The swing position which has to be taken seriously by the agenda setter is that of the crucial member of S who is least eager to replace the status quo. With this in mind, we say that player i has a Λ-*spatial swing* in winning coalition S or is Λ-*pivotal* if i has a swing in S and no other player $j \neq i$ with $d(\lambda_j, 0) \leq d(\lambda_i, 0)$, i. e. who is even less eager to replace the status quo by χ, has a swing in S. We call a player i (χ, Λ)-*pivotal* in S to abbreviate that i is Λ-pivotal in S and S is (χ, Λ)-IR. In above setting, a winning (χ, Λ)-IR coalition S has to have at least m members, and only player $(n - m + 1)$ can have a spatial swing.

To highlight the link between the spatial and the simple coalitional framework, one may define

$$\mathcal{C}_i(\chi, \Lambda) \equiv \begin{cases} \{S\} & \text{if } i \text{ has a } (\chi, \Lambda)\text{-spatial swing in } S^{9} \\ \varnothing & \text{otherwise.} \end{cases}$$

By considering all possible (χ, Λ)-combinations one then obtains the set of crucial coalitions with respect to i defined above, i. e.

$$\bigcup_{\substack{\chi \in [0, \beta], \\ \Lambda \in [\alpha, \beta]^n}} \mathcal{C}_i(\chi, \Lambda) = \mathcal{C}_i(v).$$

The refinement of swings to spatial swings captures one criterion for a crucial position to mean power in a decision framework with explicit spatial preferences. However, for a player to be truly powerful, his preferences should *matter* in terms of outcome, i. e. a small change of preferences should lead to a small change of outcome. This requires a spatial swing, but having one is not sufficient. Consider the 7-player game above and assume, for instance,

[9] Note that $S = N_{\chi \succsim 0}$ is the only (χ, Λ)-IR coalition, meaning that $\mathcal{C}_i(\chi, \Lambda)$ is well-defined.

$\lambda_A < \lambda_B < 0 < \lambda_C < \ldots < \lambda_G$,[10] i.e. only players C to G may prefer to replace the current state of affairs by some proposal $\chi \geq 0$. For $\sigma > 0$, the agenda setter wants to replace the status quo. The χ closest to his ideal point σ which establishes a (χ, Λ)-IR winning coalition S is his optimal proposal χ^*. For $0 < \sigma < \lambda_C$, $\chi^*(\sigma, \Lambda) = \sigma$ is the optimal proposal, and will become the policy outcome of the game. Player C's spatial swing position does *not* have any effect on the outcome in this case. In fact, C's preferences do not influence the outcome until $\sigma > 2\lambda_C$ holds.

Given the assumptions made for above agenda setting game, we get the following subgame perfect Nash equilibrium proposal[11]

$$\chi^*(\sigma, \Lambda) = \chi^*\left(\sigma, \lambda_{(n-m+1)}\right) = \begin{cases} \sigma & \text{if } \lambda_{(n-m+1)} \geq \frac{1}{2}\sigma \\ 2\lambda_{(n-m+1)} & \text{if } \lambda_{(n-m+1)} \in (0, \frac{1}{2}\sigma) \\ 0 & \text{if } \lambda_{(n-m+1)} \leq 0 \end{cases}$$

which is accepted by voters $(n), \ldots, (n-m+1)$ and by any $(n-m), \ldots, (l)$, $n - m \geq l \geq 1$, for whom $(\lambda_i - \chi^*)^2 \leq \lambda_i^2$ holds. Hence $\Omega^*(\sigma, \Lambda) = \chi^*\left(\sigma, \lambda_{(n-m+1)}\right)$.[12] This states more formally that, first, only the spatial swing player $(n-m+1)$ *may have* an influence on the outcome and, second, he actually *has* an influence only for particular preference constellations (here for $\lambda_{(n-m+1)} \in (0, \frac{1}{2}\sigma)$).

This calls for a further refinement of spatial swings. Namely, we say that player i has a *strict* (σ, Λ)*-spatial swing* in winning coalition S or is *strictly* (σ, Λ)*-pivotal* if his ideal policy outcome λ_i affects the agenda setter's optimal policy proposal $\chi^*(\sigma, \Lambda)$, i.e. $\partial\chi^*(\sigma, \Lambda)/\partial\lambda_i > 0$.[13] Clearly, a strict spatial swing implies a spatial swing. Note that at most one – and possibly no – voter can have a strict spatial swing for any given (σ, Λ)-realization. There can be lucky players who get more utility from the outcome than the swing

[10] Identity of two or more players' ideal points has zero probability for a continuous distribution of Λ. This case will therefore be neglected in the following.

[11] One may assume small costs of being rejected for agenda setter A to ensure uniqueness of A's proposal in the last sub-case. There are, depending on Λ, multiple subgame perfect equilibria corresponding to the same unique equilibrium proposal by agenda setter A. We focus on (χ^*, Λ)-IR coalitions.

[12] Note that the ideal point $\lambda_{(n-m+1)}$ of the pivotal player is unique. In qualified majority voting there are two potential pivotal players but agenda setting makes the equilibrium unique.

[13] The possible event for which $\chi^*(\cdot)$'s derivative is not defined has zero probability and is therefore neglected.

player. This illustrates that being powerful does not per se imply particular success.

Considering a particular (σ, Λ)-combination, the players who are not (χ, Λ)-spatially pivotal for the agenda setter's optimal proposal $\chi = \chi^*(\Lambda, \sigma)$, do never influence the policy outcome for individually rational voting. They can be compared to excess players of a winning coalition in the coalitional form framework. A player who has a spatial swing in $N_{\chi \succsim 0}$ but does not affect the agenda setter's proposal χ, i. e. has no *strict* spatial swing, is more like an inferior player in the coalitional form framework: He seems powerful as long as strategic considerations of decision-making are left out of the picture. Taking strategic interaction into account, he has no more power than true excess or dummy players.

As mentioned, very particular (σ, Λ)-combinations are of little interest for *a priori* power measurement. What matters is the a priori probability that a player ends up having power. This clearly depends on the distributional assumptions on $\tilde{\Lambda}$ and $\tilde{\sigma}$ that one makes. Forgetting for the moment the particular assumptions we have made above, it is generally useful to single out those players for which the necessary condition for influencing the outcome holds under almost no realization of ideal points, i. e. who almost never have a $(\chi^*(\sigma, \Lambda), \Lambda)$-spatial swing.

Definition 3 *A player is called* spatially dummy *if*

$$P\left\{\tilde{\lambda}_i = \tilde{\lambda}_{(n-m+1)}\right\} = 0.$$

$P\{\tilde{\lambda}_i = \tilde{\lambda}_{(n-m+1)}\} > 0$ is, however, not sufficient for a priori power in our agenda setting game. A player's spatial swing must, in addition, have positive probability of making a difference, i. e. of actually being a strict spatial swing. Player $(n - m + 1)$ has a strict spatial swing in the above setting if

$$0 < \lambda_{(n-m+1)} < \frac{1}{2}\sigma.$$

It is now in the spirit of the inferior player definition of Napel and Widgrén (2001a) to define:

Definition 4 *A player is called* spatially inferior *if*

$$P\left\{\tilde{\lambda}_i = \tilde{\lambda}_{(n-m+1)} \wedge 0 < \tilde{\lambda}_{(n-m+1)} < \frac{1}{2}\tilde{\sigma}\right\} = 0$$

The probabilistic approach to the measurement of power in coalitional form games (cp. Straffin 1977) can straightforwardly be extended to measure a priori power in voting games with random spatial preferences. Namely, one measures a player's power as the probability of having a 'powerful' position. Building immediately on the more demanding notion of power embodied by strict spatial swings, this yields:

Definition 5 *Consider a spatial voting game defined by* $(N, A, \mathcal{W}, F_{\tilde{\Lambda}}, F_{\tilde{\sigma}})$ *and agenda setting as specified above. Then, the* Strict Strategic Power Index (SSPI) ξ *is defined by* $(i \in N)$

$$\xi_i \equiv P\left\{\tilde{\lambda}_i = \tilde{\lambda}_{(n-m+1)} \wedge 0 < \tilde{\lambda}_{(n-m+1)} < \frac{1}{2}\tilde{\sigma}\right\}.$$

Recall that in coalitional form voting games, players' preferences are not explicitly modelled. It is then a standard assumption to consider any ordering of players as equally probable and to attribute a swing to the $n-m+1$-th player in a given ordering.[14] This produces the *Shapley-Shubik index (SSI)* ϕ. It may simply be expressed as

$$\phi_i \equiv P\left\{\tilde{\lambda}_i = \tilde{\lambda}_{(n-m+1)}\right\}$$

under the condition that the joint distribution of Λ makes all orderings equally probable.

The assumption of independent identically distributed (i. i. d.) uniform distributions satisfies this condition. Therefore, in the above setting, the SSPI can be expressed in terms of the Shapley-Shubik index ϕ_i in the sub-game among voters:

$$\begin{aligned} \xi_i &= P\left\{\tilde{\lambda}_i = \tilde{\lambda}_{(n-m+1)}\right\} P\left\{0 < \tilde{\lambda}_{(n-m+1)} < \frac{1}{2}\tilde{\sigma} \,\middle|\, \tilde{\lambda}_i = \tilde{\lambda}_{(n-m+1)}\right\} \\ &= \phi_i \, P\left\{0 < \tilde{\lambda}_{(n-m+1)} < \frac{1}{2}\tilde{\sigma} \,\middle|\, \tilde{\lambda}_i = \tilde{\lambda}_{(n-m+1)}\right\}. \end{aligned}$$

[14] Equivalently, the m-th player in a given order can be considered – this is just a matter of convention. A truly alternative assumption is to consider any coalition equally probable and any player in a given coalition as equally likely to leave. This leads to the *Banzhaf index.*

In order to calculate the SSPI for a given spatial voting game with i. i. d. random preferences, the following result is useful:

Lemma Consider the i. i. d. random variables $\hat{\lambda}_1, \ldots, \hat{\lambda}_n$ with density $f_{\hat{\lambda}}$ and cumulative distribution function $F_{\hat{\lambda}}$. Let $\hat{\lambda}_{(p)}$ denote the p-th order statistic of these n random variables, i. e. the (random) p-th smallest value of $\hat{\lambda}_1, \ldots, \hat{\lambda}_n$. Then

$$\begin{aligned} F_{\hat{\lambda}_i=\hat{\lambda}_{(p)}}(x) &\equiv P(\hat{\lambda}_i \leq x \wedge \hat{\lambda}_i = \hat{\lambda}_{(p)}) \\ &= \int_0^x \binom{n-1}{p-1} F_{\hat{\lambda}}(s)^{p-1} \left[1 - F_{\hat{\lambda}}(s)\right]^{n-p} f_{\hat{\lambda}}(s)\, ds. \end{aligned}$$

Proof: For both $\hat{\lambda}_i$ and $\hat{\lambda}_{(p)}$ to be equal to x, exactly $p-1$ random variables $\hat{\lambda}_j$, $j \neq i$, have to be no greater than x and the other $n-p$ random variables $\hat{\lambda}_j$, $j \neq i$, have to be no smaller than x (see e. g. Arnold et al. 1992). There are $\binom{n-1}{p-1}$ permutations of $\hat{\lambda}_j$, $j \neq i$, that satisfy this requirement. Therefore

$$\begin{aligned} &P(\hat{\lambda}_i \leq x \wedge \hat{\lambda}_i = \hat{\lambda}_{(p)}) \\ = &\int_0^x \binom{n-1}{p-1} P(\lambda_1, \ldots, \lambda_{p-1} \leq s)\, P(\lambda_{p+1}, \ldots, \lambda_n \geq s)\, f_{\hat{\lambda}}(s)\, ds \\ = &\int_0^x \binom{n-1}{p-1} F_{\hat{\lambda}}(s)^{p-1} \left[1 - F_{\hat{\lambda}(s)}\right]^{n-p} f_{\hat{\lambda}}(s)\, ds \\ = &\frac{1}{n} \underbrace{\int_0^x n \binom{n-1}{p-1} F_{\hat{\lambda}}(s)^{p-1} \left[1 - F_{\hat{\lambda}(s)}\right]^{n-p} f_{\hat{\lambda}}(s)\, ds}_{P(\hat{\lambda}_{(p)} \leq x)} .^{15} \end{aligned}$$

□

Specifically, let us consider i.i.d. U(0,1) random variables $\hat{\lambda}_1, \ldots, \hat{\lambda}_n$, their p-th order statistic $\hat{\lambda}_{(p)}$, and $\hat{\Pi}$ (independently $U(\frac{-\alpha}{(\beta-\alpha)}, \frac{\frac{1}{2}\beta-\alpha}{\beta-\alpha})$-distributed

[15] If the $\hat{\lambda}_i$ are $U(0,1)$-distributed, this means that $\hat{\lambda}_{(p)}$ is Beta-distributed with parameters $(p, n-p+1)$.

with density $f_{\hat{\Pi}}$). With this we get

$$
\begin{aligned}
& P(\hat{\lambda}_i \leq \hat{\Pi} \ \wedge \ \hat{\lambda}_i = \hat{\lambda}_{(p)}) \\
= & \int\limits_{-\infty}^{\infty} P(\hat{\lambda}_i \leq x \ \wedge \ \hat{\lambda}_i = \hat{\lambda}_{(p)})\, f_{\hat{\Pi}}(x)\, dx \\
= & \frac{1}{n} \int\limits_{-\infty}^{\infty} P(\hat{\lambda}_{(p)} \leq x)\, f_{\hat{\Pi}}(x)\, dx \\
= & \frac{1}{n} \int\limits_{-\infty}^{\frac{-\alpha}{\beta-\alpha}} 0 \cdot f_{\hat{\Pi}}(x)\, dx + \frac{1}{n} \int\limits_{\frac{-\alpha}{\beta-\alpha}}^{\frac{\frac{1}{2}\beta-\alpha}{\beta-\alpha}} P(\hat{\lambda}_{(p)} \leq x)\, f_{\hat{\Pi}}(x)\, dx + \frac{1}{n} \int\limits_{\frac{\frac{1}{2}\beta-\alpha}{\beta-\alpha}}^{\infty} 1 \cdot 0\, dx \\
= & \int\limits_{\frac{-\alpha}{\beta-\alpha}}^{\frac{\frac{1}{2}\beta-\alpha}{\beta-\alpha}} \left[\int_0^x \binom{n-1}{p-1} s^{p-1} \left[1-s\right]^{n-p} ds \right] \cdot \frac{2(\beta-\alpha)}{\beta}\, dx
\end{aligned}
$$

With $p \equiv n - m + 1$ and the above distributional assumptions, we can now derive the explicit functional form of the SSPI in our example agenda setting model:

$$
\begin{aligned}
\xi_i &= P\left\{ \tilde{\lambda}_i = \tilde{\lambda}_{(p)} \wedge 0 < \tilde{\lambda}_{(p)} < \frac{1}{2}\tilde{\sigma} \right\} \\
&= P\left\{ \tilde{\lambda}_{(p)} < \frac{1}{2}\tilde{\sigma} \ \wedge \ \tilde{\lambda}_i = \tilde{\lambda}_{(n-m+1)} \right\} - P\left\{ \tilde{\lambda}_{(p)} < 0 \ \wedge \ \tilde{\lambda}_i = \tilde{\lambda}_{(p)} \right\} \\
&= \int\limits_{-\infty}^{\infty} P\left\{ \hat{\lambda}_{(p)} < x \ \wedge \ \hat{\lambda}_i = \hat{\lambda}_{(p)} \right\} f_{\hat{\Pi}}(x)\, dx \\
&\quad - P\left\{ \hat{\lambda}_{(p)} < \frac{-\alpha}{\beta-\alpha} \ \wedge \ \hat{\lambda}_i = \hat{\lambda}_{(p)} \right\} \\
&= \int\limits_{\frac{-\alpha}{\beta-\alpha}}^{\frac{\frac{1}{2}\beta-\alpha}{\beta-\alpha}} \left[\int_0^x \binom{n-1}{p-1} s^{p-1} \left[1-s\right]^{n-p} ds \right] \cdot \frac{2(\beta-\alpha)}{\beta}\, dx \\
&\quad - \int_0^{\frac{-\alpha}{\beta-\alpha}} \binom{n-1}{p-1} s^{p-1} \left[1-s\right]^{n-p} ds.
\end{aligned}
$$

Let us finally illustrate the SSPI, and also the difference between the inferior player axiom and the spatial inferiority condition, with an example:

Example 1 *Consider the 3-person coalitional form game with* $N = \{A, B, C\}$; $\mathcal{W} = \{\{A, B\}; \{A, C\}; \{A, B, C\}\}$ *. Given a uni-dimensional policy space, a natural model for this coalitional form game is uni-dimensional* 3*-person spatial agenda setting game. Suppose that* $Q = 0$*,* $\alpha = -\frac{1}{3}$ *and* $\beta = 1$*, implying that with probability* $\frac{1}{4}$ *there is no overlap between the agenda setter's and a voter's political interests. Note that this is exactly the same example as above but modified into a spatial setting. Players* B *and* C *are inferior. Player* A *is in the position to make take-it-or-leave-it offers to them, and hence the natural agenda setter. Let us denote the ideal point of* A *by* a *and the ideal points of* B *and* C *by* b *and* c *respectively. Suppose as above that* $\tilde{a} \sim U(0, 1)$, $\tilde{b} \sim U\left(-\frac{1}{3}, 1\right)$ *and* $\tilde{c} \sim U\left(-\frac{1}{3}, 1\right)$*. Re-scaling yields* $\hat{a} \sim U\left(\frac{1}{4}, 1\right)$*,* $\hat{b} \sim U(0, 1)$*,* $\hat{c} \sim U(0, 1)$ *and* $\hat{\Pi} \sim U\left(\frac{1}{4}, \frac{5}{8}\right)$*. Note that in this simple game we have* $n = 2$ *and* $m = 1$*. This implies*

$$\begin{aligned}
\xi_B &= \xi_C = \int_{\frac{1}{4}}^{\frac{5}{8}} \left[\int_0^x \binom{1}{1} s\, ds\right] \cdot \frac{8}{3}\, dx - \int_0^{\frac{1}{4}} \binom{1}{1} s\, ds \\
&= \int_{\frac{1}{4}}^{\frac{5}{8}} \frac{8}{3} \cdot \frac{1}{2} x^2\, dx - \frac{1}{32} \\
&= \frac{4}{9}\left[\left(\frac{5}{8}\right)^3 - \left(\frac{1}{4}\right)^3\right] - \frac{1}{32} \\
&= \frac{9}{128} \approx 0.0703.
\end{aligned}$$

This is less than half of the SSI, which gives $\frac{1}{6}$ *for inferior players.*

To further illustrate the difference between the SSPI and SSI let us first remove the range in which there are no gains from "exchange", i. e. set $\alpha = 0$. This implies $\hat{a} \sim U(0, 1)$ $\hat{b} \sim U(0, 1)$, $\hat{c} \sim U(0, 1)$ and $\hat{\Pi} \sim U\left(0, \frac{1}{2}\right)$. Doing the same calculations as in the example above we get

$$\xi_B = \xi_C = \int_0^{\frac{1}{2}} \left[\int_0^x \binom{1}{1} s\, ds \right] \cdot \frac{2}{1}\, dx$$

$$= \frac{1}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{24} \approx 0.0417.$$

When it becomes more likely that a proposal is accepted, it also becomes more likely that the ideal point of the agenda setter A is accepted. Inefficiency, i. e. $\alpha < 0$, benefits the voters since it complicates strategic agenda setting. This is not the case when the agenda setter does not act strategically. Then the extent of status quo bias has no role. To see this let us assume that the agenda setter becomes like one of the voters and is acting non-strategically by always proposing $\chi = \lambda_{(n-m+1)}$.[16] In the example above, this means that the assumed agenda setter A is able to pass her ideal point in four ideal point permutations: $(\lambda_B, \lambda_A, \lambda_C), (\lambda_C, \lambda_A, \lambda_B)$, $(\lambda_B, \lambda_C, \lambda_A)$, $(\lambda_C, \lambda_B, \lambda_A)$. Note that this is independent of the value of α since in this case IR-constraints do not affect agenda setting. Players B and C are able to make a change in $(\lambda_A, \lambda_B, \lambda_C)$ and $(\lambda_A, \lambda_C, \lambda_B)$, respectively. Hence we get $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$, the SSI. The SSPI – by adding strategic agenda setting to a spatial voting model – yields something reminiscent to the SSI with the degree of similarity determined by various factors. The model demonstrates that inefficiencies in decision making, as measured by α, have significant impact on power if it is understood as the ability to make a difference.[17] Hence we get different power distributions when we let the value of α vary.[18] This is in a sense trivial: If one analyses how spatial preferences affect power, the domain and distribution of preferences matter.

[16]Alternatively we can think that the agenda setter is really like a voter and a proposal is made by an intelligent benevolent machine after the players have told it their ideal points.

[17]The values of the SSI and the SSPI are comparable as probabilities. The values of the SSPI shed some light how much difference strategic agenda setting makes to the SSI under different assumptions of the domains of preference distributions. Note, however, that the purpose of this paper is not a beauty contest between the SSI and the SSPI. Our attempt is to assess the relationship between spatial preferences and power. As a special case we get the SSI.

[18]Strictly speaking we let the ratio $\frac{\alpha}{\beta}$ vary. This ratio affects the re-scaling presented above.

3 *Concluding remarks*

In spatial voting games the individual rationality constraints above determine what kind of proposals will be accepted. Players' rates of acceptance are thus determined by the relative locations of voters' and the agenda setter's ideal points. Moreover, the agenda setter is assumed to act strategically. Strategic aspects of coalition formation were introduced into coalitional form games in Napel and Widgrén (2001a) by distinguishing between inferior and non-inferior players. The implications on players' power were discussed more in depth in Napel and Widgrén (2001b). In this paper, following this tradition, we have constructed a strategic power index, which has spatial preferences and strategic agenda setting as its main building blocks. Earlier work in this field is still preliminary. In Steunenberg et al. (1999), a different strategic power index is introduced. This measure, contrary to what we propose here, defines power as proximity between one's ideal point and the outcome of the game. But, proximity may be due to luck and, indeed, in this paper we demonstrate that under strategic agenda setting players whose ideal points are located close to the outcome tend to have luck, not power. The pivotal player is the player who exerts power, although a winner's curse often arises in terms of proximity. In fact, when the pivotal player has an effect to the outcome she gains the least among the players in a winning coalition.

In this paper, we have proposed a new strategic power index for spatial voting games. Our model has several restrictions like uni-dimensionality and a specific sequential game form. We feel, however, that we have opened an avenue for a new type of power measurement literature and further research should follow.

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