Aspiration Adaptation in the Ultimatum Minigame

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Two agents recurrently play a $2 \times 2$ version of the ultimatum game. Each player sticks to his past action if it was satisfactory relative to an endogenous aspiration level and otherwise abandons it with positive probability. This type of satisficing behavior is shown to yield efficiency in the limit. It does not favor a specific distribution of surplus and can give an explanation for the incidence of equitable offers in practice. Numerical investigations link a player’s character as captured by the model parameters to his average bargaining success. Results indicate that it is beneficial to be persistent and stubborn, i.e. slow in adapting aspirations and switching actions in response to major dissatisfaction. Also, it is an advantage to be capricious, i.e. to experience large and frequent perturbations of aspiration level and to discriminate only little between minor and major dissatisfaction.

Key Words: Bargaining, ultimatum game, bounded rationality, learning, satisficing, aspiration level, stochastic evolutionary games.

1. INTRODUCTION

Non-cooperative game-theoretic models of bargaining, such as Rubinstein (1982), traditionally assume that players are perfectly rational and that this fact is common knowledge. In order to deduce sharp game-theoretic predictions, preferences over a player’s share of surplus and time of agreement are frequently taken to be common knowledge, too. Many experimental studies of human bargaining behavior have pointed out how demanding this set of assumptions is. They indicate a need to investigate bargaining models based on more broadly defined preferences (e.g. incorporating aspects of fairness, aspirations, or history of play), and also ones dealing with boundedly rational, non-strategic players.

This paper considers repeated play of a $2 \times 2$ version of the ultimatum game, the ultimatum minigame, by two agents who use the satisficing heuristic investigated by Karandikar et al. (1998). Players stick to their past action if it performed ‘well’ relative to an endogenous aspiration level, and otherwise abandon it with positive probability. This particularly simple rule-of-thumb is analytically investigated concerning its long-run implications for efficiency and distribution. In computer simulations, the effects of soft factors—describing real-life agent characteristics like stubbornness or capriciousness—on average bargaining success are studied.

The model can be thought of as describing habitual bargaining involving e.g. two spouses or friends sharing a flat, boss and secretary, siblings, supervisor and student, or colleagues. These often trivial negotiations about the allocation of chores, how to work, what film to watch, at what time to meet, etc. may not even be recognized as a strategic game—prompting stimulus-response behavior rather than explicit calculations.

It is shown that aspiration-based satisficing behavior can provide an ex-

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2See e.g. Binmore et al. (1992) and Muthoo (1999).
planation for the observation that proposers in ultimatum bargaining frequently make equitable offers in practice. The exact way in which player’s aspirations are occasionally perturbed determines whether an inequitable division of surplus, an equitable division, or—most commonly—both are in the support of the long run outcome as identified by techniques similar to Kandori *et al.* (1993) and Young (1993a, 1993b). Inefficient disagreement has vanishing probability. The simulations confirm that equitable offers are a prominent feature of the considered dynamics. Moreover, a player turns out to benefit from stubbornness and persistence concerning the adaptation of actions and aspirations, but also from capriciousness in the sense of experiencing frequent and large perturbations of one’s aspiration level.

Section 2 introduces the model. Section 3 presents and discusses the theoretical results. In Section 4, we compare different parameter scenarios in Monte-Carlo simulations. Section 5 concerns related literature, and Section 6 concludes.

2. THE MODEL

Two players can share an available surplus provided that they agree on how to split it. Both are assumed to care only about their own share. We consider the *ultimatum minigame* (UMG) shown in Fig. 1. First, player 1 makes either a high offer, *H*, or a low offer, *L*. After *H*, acceptance is assumed and the equitable payoff vector (2, 2) results. After *L*, player 2 can either accept (action *Y*)—implying the inequitable surplus division (3, 1)—or reject (action *N*), implying that players receive (0, 0). The game may represent an entire class of situations that from a player’s point of view are equivalent in terms of payoffs and aspirations.

(*L, Y*) and (*H, N*) are the pure-strategy Nash equilibria of the UMG. The former is the unique perfect equilibrium. It corresponds to the unique asymptotically stable attractor of (unperturbed) two-population replicator dynamics (see Gale *et al.*, 1995) and is regarded as game theory’s predicted
outcome. However, the equitable distribution implied by \((H, N)\) and the maximin strategy profile \((H, Y)\) is a focal point in laboratory experiments.\(^4\)

We consider two agents who recurrently play the UMG in fixed roles. Each interaction is indexed by some \(t \in \{0, 1, 2, \ldots\}\), referred to as \textit{period} or \textit{time}. Players are assumed to use a simple \textit{satisficing heuristic}, which says:

1. Stick to your action if it was successful in the last period (compared to the aspiration level).

2. Otherwise change your action with positive probability, but not certainty.

This heuristic is myopic and involves neither backward induction by player 1 nor even simple payoff maximization by player 2.

Players’ aspirations rise after positive feedback, i.e. a payoff above the aspiration level, and fall after negative feedback. Occasionally, aspirations are perturbed. One can imagine the random change of a player’s aspiration level to result from positive or negative experience in a different (unmodeled) game, the observation of outcomes in a UMG played by dif-

\(^4\)Yang \textit{et al.} (1999), among others, confirm that behavior in the ultimatum minigame is qualitatively the same as in the ultimatum game.
ferent players, or simply an idiosyncratic optimistic or pessimistic shift in the player’s perception of the world due to the weather, last night’s sleep, etc.\textsuperscript{5} This satisficing heuristic has first been investigated by Karandikar et al. (1998) in the context of symmetric 2 × 2-games.

In any period, both players recall only their own strategy in last period’s interaction together with their personal payoff from that interaction. There is no individual memory apart from what has been ‘condensed’ into a player’s aspiration level. Agents do neither observe their opponent’s payoff, strategy, or aspiration level nor need they be aware of playing a 2-person game at all. Player 1’s state at date \( t \) refers to her action \( a_t \in A \equiv \{L, H\} \) chosen in \( t \) and her aspiration level \( \alpha_t \in A \equiv [0, 3] \) held in \( t \).\textsuperscript{6} Similarly, player 2’s state in period \( t \) is given by \( b_t \in B \equiv \{Y, N\} \) and his aspiration level \( \beta_t \in B \equiv [0, 2] \). The system’s state in \( t \) is thus a 4-tuple \( x_t = (a_t, \alpha_t, b_t, \beta_t) \), making \( E \equiv A \times A \times B \times B \) the state space. The specification of \( A \) and \( B \) reflects that players are assumed to know their maximum and minimum feasible payoff, i.e. even a perturbation cannot make player 1 aspire to more than 3. The model is suited to study the gradual evolution of aspirations, although one could also consider jumps restricted e.g. to the set \{0, 1, 2, 3\} by corresponding assumptions for aspiration adaptation and perturbations.

Player 1’s actions are updated as follows. The probability that she sticks to \( a_t \) in \( t + 1 \) is a (weakly) decreasing function \( p_1 \) of her disappointment, \( \Delta_1 \equiv \alpha_t - \pi_1(a_t, b_t) \), with \( a_t \) in period \( t \)’s interaction as defined by actual and aspired payoffs (\( \Delta_1 \) is non-positive if payoff was actually satisfactory). She repeats \( a_t \) with positive probability even if she is maximally dissatisfied, i.e. \( p_1(\Delta_1) \) is bounded below by some \( \tilde{p}_1 \in (0, 1) \). This reflects inertia

\textsuperscript{5}Under certain constraints, perturbations of aspirations can indirectly model perturbations of payoffs or inertia.

\textsuperscript{6}Some may prefer to call \( a_t \) player 1’s behavioral mode to stress that she does not ‘choose’ in the traditional economic way.
her behavior, also interpreted as stubbornness. Action $a_t$ is with certainty repeated if she was satisfied, i.e. $p_1(\Delta_1) = 1$ for $\Delta_1 \leq 0$.

Function $p_1$ is assumed to be continuous. Moreover, the rate at which it falls is taken to be both bounded from above and away from zero. These assumptions formalize that willingness to switch behavior is changing gradually and at moderate rate. Moreover, player 1’s propensity to stick with the past action decreases visibly as soon as she is dissatisfied.

The assumptions for player 2 are analogous, with function $p_2$ bounded below by $\tilde{p}_2 \in (0,1)$. We will refer to functions $p_i$ as inertia functions and drop the player subscript on $\Delta$. A possible inertia function $p_i$ is illustrated in Fig. 2. Action updating is assumed to be stochastically independent across time and players for any given state.

Next, consider the updating of aspirations. We distinguish two update rules. The first defines a Markov process referred to as the unperturbed satisficing process $\Phi^0$. In $\Phi^0$, a player’s aspiration level in $t+1$ is simply a weighted average of the aspiration level in $t$ and the payoff received in $t$.

\[ \text{FIG. 2 A possible inertia function } p_i \]

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More precisely,

\[ \alpha_{t+1} = \alpha^*(x_t) \equiv \lambda \alpha_t + (1 - \lambda) \pi_1(a_t, b_t) \tag{1} \]

with \( \lambda \in [0, 1] \) for player 1. The larger \( \lambda \), the slower aspirations change—for \( \lambda = 1 \) initial aspiration \( \alpha_0 \) is kept forever, for \( \lambda = 0 \) aspirations equal the most recent payoff experience. Parameter \( \lambda \) measures the persistence of player 1. Player 2’s aspirations are exponentially smoothed analogously and, for the time being, with identical parameter \( \lambda \).

An alternative update rule defines the perturbed satisficing process \( \Phi^\eta \) (\( \eta > 0 \)). First, players’ aspirations are deterministically updated as in \( \Phi^0 \). Second, each player’s aspiration level is independently perturbed with probability \( \eta \). The new aspiration level is a random variable whose distribution can depend on the deterministically updated ‘intermediate’ aspiration level \( \alpha^*(x_t) \) or \( \beta^*(x_t) \), respectively.

To facilitate theoretical analysis, we assume both players to have the same state-independent probability \( \eta > 0 \) of experiencing a tremble. Post-perturbation aspirations for players 1 and 2 are assumed to have the distinct, state-dependent, and continuous conditional densities \( g_1(\cdot|\alpha^*(x_t)) \) and \( g_2(\cdot|\beta^*(x_t)) \) with supports contained in \( A \) and \( B \), respectively. For any interval around \( \alpha^*(x_t) \), let the probability that \( \alpha_{t+1} \) lies in it be positive, i.e. shocks can be small. Also, it is convenient to assume that \( \alpha_{t+1} \) lies in a neighborhood of \( \alpha_t \) with positive probability, i.e. player 1 can ‘forget’ her most recent payoff experience as the result of a perturbation.\(^8\) Similar assumptions apply for density \( g_2 \).

3. THEORETICAL RESULTS

We first investigate the unperturbed satisficing process \( \Phi^0 \). Let us call a state in which both players’ aspiration levels equal the respective payoffs

\(^8\)These assumptions imply that \( \Phi^\eta \) has the quite intuitive property of open-set irreducibility. \( \psi \)-irreducibility is, however, sufficient for the results.
a *convention*. This views a convention as a regularity of behavior to which both players conform, and which no player is prompted to abandon given that the other one conforms (cf. Lewis, 1969). We write $c_{ab}$ for convention $(a, \pi_1(a,b), b, \pi_2(a,b))$. The UMG has four conventions, collected in the set $C$. As $c_{HY}$ and $c_{HN}$ yield the same surplus distribution, they are typically not distinguished; $c_H$ refers to their union. In the following, we examine whether the conventions are stable under the specified satisficing dynamics. A preliminary result is:

**Proposition 1.** State $x \in E$ is an absorbing state of $\Phi^0$ if and only if it is a convention. For $\lambda < 1$ and any initial state $x_0 \in E$, $\Phi^0$ converges almost surely to a convention $c \in C$.

The first part follows from $p_i(0) = 1$ and the aspiration update rule (1). Proof of the second part amounts to confirming that from any state $x_t \notin C$ there is a positive probability $\varepsilon > 0$ of starting an infinite run on the present action pair in period $t$ (see Karandikar et al., 1998, Proposition 1).

The proposition states that players’ adaptive interaction in $\Phi^0$ will eventually establish a convention, but any of the three possible bargaining results—symmetric or equitable division, asymmetric or inequitable division, and break-down—can, depending on the initial state, be selected. In contrast, if players experience trembles in their aspiration levels, the influence of the initial state $x_0$ is gradually washed away. $\Phi^\eta$ is an *ergodic* Markov process:

**Proposition 2.** For $\lambda < 1$ and any given perturbation parameter $\eta \in (0,1]$, $\Phi^\eta$ converges (strongly) to a unique limit distribution $\mu^\eta$ which is independent of the initial state $x_0 \in E$.

The proof is given in the appendix. Proposition 2 establishes that $\Phi^\eta$’s long-run behavior is accurately described by a stationary distribution $\mu^\eta$. In particular, the empirical frequency distribution over states up to a period $t$, sampled from an arbitrary process realization, converges to $\mu^\eta$ as $t \to \infty$. 
It is generally not possible to give details on $\mu^\eta$ for arbitrary $\eta$ and $\lambda$. Our analytical investigation concentrates on the benchmark case in which the probability of a tremble, $\eta$, is close to zero and in which, additionally, present payoff experience affects players’ aspirations only marginally, i.e. $\lambda$ is close to one.\textsuperscript{9} Our first main result is

**Theorem 1.** Let the perturbed satisficing process $\Phi^\eta$ be defined as above and $\lambda < 1$. The limit stationary distribution of $\Phi^\eta$ for $\eta \to 0$, $\mu^*$, can place positive weight only on the Pareto-efficient conventions $c_{LY}$ and $c_{H}$ as $\lambda \to 1$.

The proof is given in the appendix. Theorem 1 establishes that the players will (approximately) divide the available surplus efficiently if aspiration trembles are rare and aspirations are updated slowly. The intuition for this is the following: as the probability of a tremble approaches zero, $\Phi^\eta$ becomes more and more like the unperturbed process $\Phi^0$. So by Proposition 1, $\Phi^\eta$ will spend most time in a convention.\textsuperscript{10} However, the inefficient convention $c_{LN}$ is unstable: A single perturbation of one player’s aspiration level leads to convention $c_{H}$ or $c_{LY}$ with positive probability because a likely action switch by the now dissatisfied player results in an efficient bargaining outcome that satisfies both players. The strategy combination is repeated forever unless another perturbation results in some player’s dissatisfaction. Then, however, a move to $c_{LN}$ is unlikely even if the perturbation results in the play of $(L, N)$: For $\lambda$ close to 1, an enormous number of periods has to pass before players’ aspiration are close to $(0, 0)$. The odds are that at least one of the two dissatisfied players switches action before that happens, setting the course towards an efficient convention again. Our second main result is

**Theorem 2.** Let the perturbed satisficing process $\Phi^\eta$ be defined as above,

\textsuperscript{9}This investigation is the main reason for working with a continuous state space.

\textsuperscript{10}Here, we implicitly refer to an arbitrary neighborhood of the respective state $c_{ab}$. 

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and let $\mu^*$ denote the limit stationary distribution of $\Phi^\eta$ for $\eta \to 0$. The supports of post-perturbation aspiration densities in conventions $c_H$ and $c_{LY}$ can be chosen

i) such that $\mu^*$ places all weight on the asymmetric efficient convention $c_{LY}$,

ii) such that $\mu^*$ places all weight on the symmetric efficient convention $c_H$, or

iii) such that $\mu^*$ places positive weight on both $c_{LY}$ and $c_H$,

as $\lambda \to 1$. If, in particular, $g_i(\cdot|c_{LY})$ and $g_i(\cdot|c_H)$ have full support in $A$ and $B$, both $c_{LY}$ and $c_H$ have positive weight.

The proof is given in the appendix. Theorem 2 implies that the average surplus distribution selected by the satisficing heuristic depends on the distribution of aspiration perturbations even for $\eta \to 0$ and $\lambda \to 1$. There is nothing in theory which discriminates in favor of the perfect equilibrium outcome. The intuition for this result again rests on the approximative description of $\Phi^\eta$ as the composition of the unperturbed satisficing process $\Phi^0$ and rare interruptions by a perturbation of one player’s aspiration level. By Theorem 1, only perturbations occurring in $c_{LY}$ and $c_H$ matter. There exist small neighborhoods $U(c)$ of conventions $c \in \{c_{LY}, c_H\}$ such that a perturbation into $U(c)$ leads to $c$ with arbitrarily higher probability than to convention $c' \neq c$. So, if perturbations from $c_{LY}$ are never leading to aspirations outside $U(c_{LY})$ and if, in contrast, there is positive probability of a perturbation from $c_H$ into $U(c_{LY})$, then $\Phi^\eta$ will spend almost all time in the asymmetric convention $c_{LY}$. Similarly, the invariant distribution is as in ii) when perturbations from $c_H$ are very ‘narrow’ and those from $c_{LY}$ are ‘wide’. If perturbations have reasonably wide or even full support in both $c_{LY}$ and $c_H$, iii) applies.

The stationary distribution of an ergodic stochastic process captures average behavior as time goes to infinity. Considering the limit of such
a limit distribution as $\eta \to 0$ and as $\lambda \to 1$, i.e. trembles vanish and persistence of once formed aspirations becomes large, is a useful benchmark. Still, in actual process realizations with parameters close to the limit, the Pareto-inefficient outcome excluded by Theorem 1 will be observed. With this caveat, one may summarize above mathematical results as a quite robust prediction in terms of efficiency but not of a specific distribution of surplus.

4. SIMULATION RESULTS

It is worthwhile to investigate whether Theorems 1 and 2 in fact provide a good benchmark if parameters $\eta$ and $\lambda$ are not yet close to their respective limit. Is there a bias in favor of the perfect equilibrium $(L, Y)$? Moreover, how parameter-sensitive is the average surplus distribution and do monotonic trends exist? We look at a number of parameter scenarios using Monte-Carlo simulation.\footnote{Linear algebra methods were applied for control purposes, and Fig. 4 is actually based on a $31 \times 21$ grid approximation of $E$ and the left-eigenvector of a sampled transition matrix.}

We assume piece-wise linear inertia functions

$$p_i(\Delta) = \left\{ \begin{array}{ll} 1; & \Delta \leq 0 \\ 1 - M_i \Delta; & \Delta \in (0, \frac{1 - \tilde{p}_i}{M_i}) \\ \tilde{p}_i; & \Delta \geq \frac{1 - \tilde{p}_i}{M_i} \end{array} \right.$$  

with parameters $\tilde{p}_i, M_i \in \mathbb{R}^+ (i = 1, 2)$, and truncated normal perturbation distributions with means $\alpha^*(x_t)$ and $\beta^*(x_t)$ and player-specific standard deviations $\sigma_1$ and $\sigma_2$, respectively.\footnote{This implies that case iii) of Theorem 2 applies. Inertia functions which fall geometrically or uniform distributions with reasonably wide supports produce qualitatively the same observations.} For more flexibility in modeling different agent characters we drop the requirement that both players have the same aspiration persistence and the same perturbation probability.
FIG. 3 Aspiration movements as $c_H$ is challenged and replaced by $c_{LY}$

FIG. 4 Marginal stationary distribution over aspirations in scenario $S_0$

The reference scenario $S_0$ is based on the following parameter choices: $\eta = (0.05, 0.05)$, $\lambda = (0.8, 0.8)$, $\tilde{p} = (0.7, 0.7)$, $M = (1.0, 1.0)$, and $\sigma = (0.1, 0.1)$. In $S_0$, both players on average experience one perturbation in 20 updates, lower their aspiration by 20% after one period of disagreement, and stick to their action with at least probability 0.7.

Typical dynamics of $\Phi^0$ are characterized by long stretches of time spent in one efficient convention, which is occasionally challenged and eventually replaced by the other efficient convention (cf. Fig. 3). So, the perturbed satisficing process is characterized by punctuated equilibria. An approximation of the marginal stationary distribution over aspirations for scenario $S_0$ is depicted in Fig. 4, corresponding to the following approximate marginal...
distribution over action pairs:\textsuperscript{13}

<table>
<thead>
<tr>
<th></th>
<th>$LY$</th>
<th>$LN$</th>
<th>$HY$</th>
<th>$HN$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Prob}(a,b)$</td>
<td>0.367</td>
<td>0.015</td>
<td>0.185</td>
<td>0.433</td>
</tr>
</tbody>
</table>

Aspirations are already very concentrated at levels corresponding to the efficient conventions. The inequitable perfect equilibrium $(L,Y)$ is not favored over strategy combinations $(H,\cdot)$. The Nash equilibrium $(H,N)$ is more than double as frequent as non-Nash equilibrium $(H,Y)$. Latter observation is, however, not robust to parameter variations. In contrast, the observation of only little $(L,N)$-play even when $\eta$ and $\lambda$ are quite distant from 0 and 1, respectively, is robust. Therefore, one can concentrate on average payoff of player 1, denoted by $\bar{\pi}_1$, in the following sensitivity analysis; the average payoff of player 2 is slightly less than $4 - \bar{\pi}_1$ and comparative statics for $\bar{\pi}_2$ can be inferred from the $\bar{\pi}_1$-diagram.

We first vary $\tilde{p}_1$ and $\tilde{p}_2$. Given the moderately large values of slope $M_i$, these lower bounds on inertia are the value of $p_i$ within region $R_I$ of aspiration space, the open rectangle defined by corner points $(2,2)$ and $(3,1)$ (see Fig. 3), when the present action pair is $(L,N)$. In this conflict region no strategy pair simultaneously satisfies players 1 and 2, so that inertia has the effect of stamina after $(L,N)$-play: Whoever loses patience and switches action first strongly increases the chances of eventual convergence to the less-favored efficient convention. With the caveat that Fig. 5 gives ceteris paribus information,\textsuperscript{14} i.e. all parameters except $\tilde{p}_1$ and $\tilde{p}_2$ are as in scenario S0, we can state:

Observation 1. Player $i$’s average bargaining success increases with his minimal level of action inertia, $\tilde{p}_i$.

\textsuperscript{13}The approximation is obtained by a long-run simulation of 20m periods.

\textsuperscript{14}For each $(\tilde{p}_1,\tilde{p}_2)$-combination, the stationary distribution over actions was approximated at least by 10m periods of Monte-Carlo simulation. The same holds for the numbers depicted in figures 6–9.
Loosely speaking and everything else being equal, it pays to be stubborn or persistent after major dissatisfaction. However, the sum of both players’ average payoffs decreases due to more (L, N)-play when both $\tilde{p}_1$ and $\tilde{p}_2$ are increased (see Fig. 6).

The second varied parameter, slope $M_i$, defines how drastic player $i$’s response to minor dissatisfaction is. $M_1$ is relevant especially when $\alpha_t$ is slightly above 2 and $(H, L)$ or $(H, N)$ has been played. One may interpret $M_i$ as a parameter representing a player’s irritability. Figure 7 then indicates that from a boundedly-rational bargaining perspective it is (weakly) beneficial to be more irritable. More formally stated (with the caveat above):
Observation 2. Player $i$‘s average bargaining success increases with the slope $M_i$ of her inertia function.

So, although it pays to be stubborn in response to major dissatisfaction, it is beneficial to be comparatively quickly agitated by minor frustration. Greater irritability translates into greater propensity to pick a ‘fight’ which challenges the less-preferred convention after a small upward tremble in aspiration. Also, it results in fiercer resistance when the latter is approached from inside $R_I$. Next, aspiration persistence parameter $\lambda_i$ is varied. We summarize Fig. 8 (with the above caveat):

Observation 3. Player $i$‘s average bargaining success increases with the
Again, one may in more colloquial terms infer that it pays to be stubborn or persistent—this time referring to aspiration rather than action updating. The intuition for this is that in the critical conflict region of aspiration space, $R_I$, a greater $\lambda_1$ decelerates player 1’s moves towards ‘surrender’ (region $R_{II}$) but does not affect moves towards ‘victory’ (region $R_{IV}$). If one imagines an (unmodeled) encompassing biological or social evolution of agent characters, Observation 3 provides some justification for consideration of the limit case $\lambda \to 1$.

Figures 9 and 10 concern the frequency of perturbations, $\eta_i$, and the standard deviation, $\sigma_i$, of the normal distributions which define post-perturbation aspirations. Results can be summarized as follows:

**Observation 4.** Player $i$’s average bargaining success increases with the probability, $\eta_i$, of trembles in his aspirations.

**Observation 5.** Player $i$’s average bargaining success increases with the standard deviation, $\sigma_i$, of her aspiration perturbations.

A player obtains a bigger average share of the cake when she is capricious, i.e. has frequent and large variations in her mood or perception of the world. An intuition for this can be found in the asymmetric effect of upward
and downward perturbations. An upward tremble in player 1’s aspirations when her less-preferred convention $c_H$ is in place results in frustration, which will typically lead to an action switch to $L$. This bears the chance to establish 1’s most-preferred convention $c_{LY}$ for a long time. In contrast, a downward tremble in player 1’s aspirations while $c_{LY}$ is in place generally remains unnoticed and, hence, unexploited by player 2.\textsuperscript{15}

Observation 4 raises a question if one again imagines an unmodeled encompassing social or biological evolution of player characteristics: If it is advantageous to have a high propensity for spontaneous shifts in aspiration, can the limit case $\eta \to 0$ be expected to have much practical relevance? A proper answer—related to the general relevance of the limit investigations in the tradition of Kandori \textit{et al.} (1993) and Young (1993a)—cannot be given in this paper.

5. RELATION TO THE LITERATURE

To our knowledge, above model is the first to combine stochastic evolutionary methodology with satisficing behavior in a bargaining situation. We draw on the work of Karandikar \textit{et al.} (1998), who consider symmetric

\textsuperscript{15}This indicates that it is beneficial to experience asymmetric trembles.
$2 \times 2$-games with only one efficient outcome. We find satisficing based on endogenous aspiration levels sufficient for approximate long-run efficiency even in the case of competing efficient conventions.

The assumption of satisficing with endogenous aspiration levels has a long tradition (Simon, 1955 and 1959; Sauermann and Selten, 1962; Selten, 1998). Gilboa and Schmeidler (1996) show that it can, in fact, be a viable optimization heuristic with very low information processing and calculation requirements, though this is put into a more sceptical perspective by Börgers and Sarin (2000). Different versions of satisficing based on endogenous aspiration levels in an interactive setting have been investigated by Dixon (2000), Pazgal (1997), and Kim (1999). They are concerned with common interest games, and findings are similar to those of Karandikar et al. (see Bendor et al., 2001).

The investigation of satisficing players is still rather rare in the stochastic evolutionary literature. Most authors favor dynamics based on myopic best-replies (Kandori et al. 1993; Young, 1993a and 1993b), imitation (Robson and Vega-Redondo, 1996; Josephson and Matros 2000; Hehenkamp, 2002), or possibly a combination (e.g. Kaarboe and Tieman, 1999). Myopic best-response behavior is already ‘too rational’ to support $(H, \cdot)$ as a stable outcome. This seems different when imitation dynamics are considered. For these and satisficing dynamics based on random exogenous aspiration levels, approximation e.g. by replicator dynamics is possible for finite time horizon (see Benaim and Weibull, 2000). Explicit satisficing models are, nevertheless, worthwhile since predictions concerning asymptotic behavior can differ significantly (cf. Börgers and Sarin, 1997).

Gale et al. (1995) study the UMG in a replicator framework.\footnote{They consider infinite populations and expected drift in deterministic replicator equations.} $(L, Y)$ is an asymptotically stable equilibrium. Only if noise in the responder population is sufficiently higher than in the proposer population (which can be
motivated), a mixed responder population combined with $H$-offers by—in the limit—all proposers also becomes asymptotically stable. In our model, the specification of noise is crucial, too. First, supports of perturbation densities in the efficient conventions qualitatively determine long-run behavior. Second, Gale et al.’s finding that more noisy behavior can benefit a population is confirmed by our comparative statics. In contrast to Gale et al., a symmetric division can (in the limit) be the unique long-run outcome independent of the initial state in our model.

Other adaptive learning or evolutionary models of ultimatum bargaining which support the symmetric surplus division include Roth and Erev (1995), Harms (1997), Huck and Oechssler (1999), Peters (2000), and Poulsen (2000). A symmetric division also features prominently in Ellingsen’s (1997) analysis of Nash demand bargaining. In contrast, the computer simulations by van Bragt et al. (2000) clearly favor the asymmetric perfect equilibrium division.

Shor (2000) considers experimental data on monopolistic price setting in a low-information environment. Among the several learning heuristics investigated in the literature, he finds greatest support for satisficing behavior as assumed by Karandikar et al. and in this paper. In experiments on continuous-time Cournot competition with limited information, Friedman et al. (2001) confirm the advantage of being a slow mover or, in our model interpretation, of being persistent and stubborn.

Bergin and Lipman (1996) warn that any invariant distribution of an unperturbed adaptation process can be ‘selected’ by the perturbed process if perturbation rates are state-dependently chosen in an appropriate way. State-dependent perturbation supports, in fact, drive Theorem 2, but a highly implausible type of state-dependent rates would be required to ensure positive weight on the inefficient outcome in the limit.
6. CONCLUDING REMARKS

The first of our main findings is that aspiration-based satisficing behavior—quite remote from traditional economic rationality at the individual level—suffices to reach approximate efficiency in the simple UMG bargaining situation, i.e. despite the conflict of interests regarding more than one efficient outcome. The predominance of either equitable or inequitable surplus division depends on player characteristics usually not considered in bargaining models. In particular, it turned out to be, ceteris paribus, beneficial to be persistent when it comes to updating actions and aspirations, but nevertheless to be capricious. These observations seem broadly consistent with anecdotal evidence about real-life bargaining situations e.g. among colleagues, friends, or family. Also, the incidence of equitable offers by the proposer is in line with observations in bargaining experiments.

For proper statements about the practical relevance of our very weak behavioral axioms and their implications, more experiments in the vein of Lant (1992), Mookherjee and Sopher (1994, 1997), Slembeck (1999), or Binmore et al. (2002), and simulations with actual experimental data like Roth and Erev (1995) and Shor (2000) seem worthwhile. The property of endogenous transitions between different adaptive equilibria or conventions could be tested in long-run experiments. The comparative statics observations could possibly be tested by letting human players interact repeatedly with each other and also machine players of well-specified characteristics. If the inherent exploitability of machine players’ satisficing behavior were discovered by the human agents, or the comparative statics contradicted above results, the ‘lower bound’ on adaptive human bargaining behavior given in this paper would need to be raised.

Changes to above aspiration adaptation rule, incorporating e.g. overshooting, time lags, or different forms of randomness, would in most cases leave the analytical results in place. Similarly, little would change if agents
‘trembled’ in implementing their actions.\textsuperscript{17}

Extending the model to bigger strategy sets—for example to improve the discrete approximation of the ultimatum game or to allow for alternating offers—is an interesting option. Unfortunately, analytical investigation gets considerably more complicated. It would probably be more worthwhile to rather consider more than two players (see Dixon 2000). This would allow for a shock model in which players’ aspirations are also shifted by occasional observations of the success of other players. In a population-based model, it would then be natural to investigate the effects of different interaction structures (see e. g. Ellison, 1993; Berninghaus and Schwalbe, 1996; Tieman \textit{et al.}, 2000).

APPENDIX: PROOFS

The proofs draw on techniques used by Karandikar \textit{et al.} (1998), which are in turn based on Meyn and Tweedie (1993). Details are given in Napel (2000).

\textit{Proof of Proposition 2.} Let $\sigma(E)$ denote the Borel $\sigma$-algebra on $E$. The unperturbed process $\Phi^0$ is defined by a transition kernel $P : E \times \sigma(E) \to [0, 1]$, where $P(x_t, S)$ is the probability to reach the set $S$ from state $x_t$ in period $t+1$. Four transitions to singleton sets can have positive probability for $\Phi^0$—namely those to $x_{t+1} = (a, \alpha^*(x_t), b, \beta^*(x_t))$, where feasible one-step transitions in action space are indicated in Fig. 11 (see Fig. 3 for the definition of the open rectangles $R_I-R_{IV}$).

Let $Q_i$ denote the kernel of the perturbed satisficing process $\Phi^\eta$ conditioned on the event that only player $i$’s aspirations are perturbed. $Q \equiv 1/2(Q_1 + Q_2)$ denotes $\Phi^\eta$’s transition rule conditioned on exactly one player experiencing a tremble with both players equally likely to be the one. With

\textsuperscript{17}Technically, perturbation of aspirations would still be needed to ensure $\Phi^\eta$’s ergodicity.
FIG. 11 Illustration of action updating in the four regions of aspiration space

$Q_*$ as the transition probability kernel conditioned on both players’ aspirations being perturbed, the complete kernel $P^\eta$ for $\Phi^\eta$ can be written as

$$P^\eta = (1 - \eta)^2 P + 2\eta(1 - \eta)Q + \eta^2 Q_*.$$

$Q_*$ is lower semi-continuous (l.s.c.) in $x$ or a strong Feller kernel. For all $x \in E$, $S \in \sigma(E)$ we have $P^\eta(x, S) \geq \eta^2 Q_*(x, S)$. This establishes that $\Phi^\eta$ is a $T$-chain (see Meyn and Tweedie, 1993, p. 127).

$P^\eta$ is open set irreducible, i.e. every open neighborhood $U_x$ of every $x \in E$ can be reached from any $y \in E$ in a finite number of transitions. This follows from $A \times B$’s compactness, our assumptions on the supports of perturbation densities, $g_1$ and $g_2$, and the fact that any action profile can be reached with positive probability in at most three steps in $R_I$ (and then be preserved by players’ inertia).

Moreover, $\Phi^\eta$ is strongly aperiodic: First, given the inertia in action choices, the supports of perturbation densities $g_1$ and $g_2$ define, e.g. for $x^* \equiv c_{HN}$, a neighborhood $U_{x^*}$ such that $P^\eta(x, U_{x^*}) \geq \nu_1(U_{x^*}) > 0$ for any $x \in U_{x^*}$ and some function $\nu_1$. Second, from any $x \in E$ one can reach $U_{x^*}$ with positive probability in a finite number of steps. The proposition
then follows from Theorem 16.2.5 in Meyn and Tweedie (1993, p. 395).

We consider a few lemmata before turning to Theorems 1 and 2. First, define the kernel \( R : E \times \sigma(E) \to [0,1] \) with

\[
R(x,S) \equiv \lim_{n \to \infty} P^n(x,S), \quad x \in E, \ S \in \sigma(E),
\]

where \( P^n \) denotes the \( n \)-step transition kernel of \( \Phi^0 \) inductively defined from \( P \). By Proposition 1 the limit exists. \( R \) defines a ‘fast-forward’ Markov process which moves from \( x \) directly to a convention with probabilities for each \( c \in C \) defined by \( \Phi^0 \)’s long-run behavior. Second, define an artificial cousin, \( \Theta \), of the perturbed process \( \Phi^\eta \) by considering the result of just one perturbation by exactly one player—a transition according to \( Q \)—and of running the unperturbed process—described by \( R \)—for evermore afterwards. \( \Theta \) has the transition kernel \( QR \) with

\[
QR (x,S) \equiv Q(x,\cdot)R (S) = \int Q(x,ds)R(s,S)
\]

for \( x \in E \) and \( S \in \sigma(E) \). The long-run behavior of \( \Phi^\eta \) for \( \eta \) close to zero is very similar to that of \( \Theta \) and, in fact, one has:

**Lemma 1.** The sequence \( \{\mu^\eta\}_{\eta \in (0,1]} \) of limit distributions of \( \Phi^\eta \) converges weakly to a unique distribution \( \mu^* \) on \((E,\sigma(E))\) as \( \eta \to 0 \). \( \mu^* \) coincides with the unique invariant probability measure of \( \Theta \).

**Proof.** The result follows directly from Theorem 2 in Karandikar et al. (1998), where it remains to check that \( QR \) has a unique invariant measure. By Proposition 1 any \( c \in C \) is an absorbing state for transitions according to \( R \) and a transition according to \( R \) always results in some \( c \in C \). Starting in \( c_{LN} \), a transition according to \( Q \) leads into \( R_{III} \) with positive probability, and from there a transition to any \( c \in C \) has positive probability according to \( R \). Starting in \( c_{H} \) or \( c_{LY} \), a transition according to \( Q \) leads into \( R_{I} \) with positive probability, and from there a transition to any \( c \in C \) has positive probability according to \( R \). Hence, \( \Theta \) is \( \psi \)-irreducible and recurrent. This
implies that $\Theta$ has a unique invariant measure (cf. Meyn and Tweedie, pp. 133 and 182, and Theorem 10.4.9).

First, we investigate unperturbed satisficing dynamics, $\Phi^0$, in region $R_{IV}$. For $\hat{\alpha}, \hat{\beta} \in (0, \frac{1}{2})$, let $I(\hat{\alpha}, \hat{\beta}) \equiv [2 + \hat{\alpha}, 3 - \hat{\alpha}] \times [\hat{\beta}, 1 - \hat{\beta}]$ be a rectangle in $R_{IV}$. One can establish a lower bound on the number of periods that are needed to exit the larger rectangle $I(\hat{\alpha}_2, \hat{\beta}_2)$ from anywhere in $I(\hat{\alpha}, \hat{\beta})$. This number goes to infinity as $\lambda$ approaches 1. Therefore the probability of not playing the only mutually satisfying action pair $(L,Y)$ at least once before $R_{IV}$ is left can be made arbitrarily small. However, once $(L,Y)$ is played with aspirations in $R_{IV}$, $\Phi^0$ converges to $c_{LY}$. This yields:

**Lemma 2.** Given $\hat{\alpha}, \hat{\beta} \in (0, \frac{1}{2})$ and any $\varepsilon > 0$, there exists $\lambda_1 \in (0,1)$ such that

$$\lambda \operatorname{Prob}(x_t \rightarrow c_{LY} \mid x_t = (a_t, \alpha_t, b_t, \beta_t) \land (\alpha_t, \beta_t) \in I(\hat{\alpha}, \hat{\beta})) > 1 - \varepsilon$$

for all $\lambda \in (\lambda_1, 1)$ and all $t \geq 1$ in unperturbed process $\Phi^0$.

An analogous result holds for dynamics in region $R_{II}$, with rectangles $J(\hat{\alpha}, \hat{\beta}) \equiv [\hat{\alpha}, 2 - \hat{\alpha}] \times [1 + \hat{\beta}, 2 - \hat{\beta}]$ for $\hat{\beta} \in (0, \frac{1}{2})$ and convergence to $c_{H}$.

Next, consider region $R_{III}$: For $\lambda \rightarrow 1$, the number of $(L,N)$-plays needed to lower a player’s aspirations from given $v$ to $u$ increases without bound. The probability that player 1 (2) does not switch to the satisfying $H (Y)$ for all these periods vanishes. This is the reasoning behind the following lemma, and an obvious analogue concerning player 2’s aspirations:

**Lemma 3.** Given positive numbers $u$ and $v$ with $u < w \equiv \min\{2, v\}$ and any $\varepsilon > 0$, there exists $\lambda_2 \in (0,1)$ such that for arbitrary date $T$

$$\lambda \operatorname{Prob}(\alpha_t < u \text{ for some } t \geq T \mid \alpha_T \geq v) < \varepsilon$$

for all $\lambda \in (\lambda_2, 1)$ in unperturbed process $\Phi^0$. 

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Proof of Theorem 1. By Proposition 1, kernel $R$ concentrates all weight on the conventions, and the same must be true for $QR$. Then, by Lemma 1, $\mu^*$ must place zero weight on all states except $c_{LY}$, $c_{LN}$, and $c_{H}$. All aspects of $QR$ relevant to asymptotic behavior can be captured by a $3 \times 3$-matrix $S = (s_{jk})$ where cell $s_{jk}$ contains the probability of a transition from $c_j$ to $c_k$ ($j, k \in \{LY, LN, H\}$). This matrix depends on parameter $\lambda$.

Given an arbitrary $\lambda < 1$ one can find $\varepsilon(\lambda) > 0$ such that $s_{LY,LN} < \varepsilon(\lambda)$ and $s_{H, LN} < \varepsilon(\lambda)$ by Lemma 3 and its analogue for player 2, and $\varepsilon(\lambda) \to 0$ as $\lambda \to 1$. In contrast, given an arbitrary $\lambda < 1$ there exists $\delta(\lambda) > 0$ such that $s_{LN,LY} + s_{LN,H} \geq \delta(\lambda)$. This uses that perturbation densities $g_1(\cdot|x)$ and $g_2(\cdot|x)$ have support in a non-degenerate neighborhood of $c_{LN}$. Lemmata 2 and 3 (and their analogues for $R_{II}$ and player 2, respectively) imply that $\delta(\lambda) \to 1$ as $\lambda \to 1$. This establishes Theorem 1.

Lemma 4. Given arbitrary $(\alpha, \beta) \in R_I$ and $\varepsilon > 0$, there exists $\lambda_3$ such that for arbitrary date $T \geq 1$

\[ \lambda \text{Prob}((\alpha_t, \beta_t) \in R_{II} \cup R_{IV} \text{ for some } t > T \mid x_T = (a_t, \alpha, b_t, \beta)) > 1 - \varepsilon \]

for all $\lambda \in (\lambda_3, 1)$ in unperturbed satisficing process $\Phi^0$.

Proof. Consider arbitrary but fixed $(\alpha, \beta) \in R_I$. Any direct move from $R_I$ into $R_{III}$ will result in a move back to $R_I$ or to $R_{II} \cup R_{IV}$ with arbitrarily high probability by Lemma 3 and its analogue for player 2. The probability of staying in $R_I$ for ever without convergence to a convention is zero by Proposition 1. However, the probability of an infinite number of $(L,Y)$-plays despite dissatisfaction of player 2, which would be necessary to reach $c_{LY}$ from inside $R_I$, is no more than

\[ \prod_{t=1}^{\infty} p_2(\lambda^t(\beta - 1)) , \]

which approaches zero for $\lambda \to 1$. In particular, we can find $\lambda'$ such that above probability is less than $\varepsilon/2$. Similarly, there is $\lambda''$ such that the
upper bound on the probability of an infinite number of $(H,\cdot)$-plays despite
dissatisfaction of player 1 is smaller than $\varepsilon/2$.

Now, for arbitrary $\lambda_4 \in (0, 1)$ and a fixed aspiration level of player 1 in $R_I$, $\alpha \in (2, 3)$, we define

$$\hat{p}_{(L,\cdot)}(\alpha) \equiv \sup_{\lambda \in (\lambda_4, 1)} \sup_{\beta \in (1, 2)} \max_{(a,b) \in A \times B} \lambda \operatorname{Prob}(a_{t+2} = L \mid x_t = (a, \alpha, b, \beta))$$

to establish an upper bound on the probability that player 1 will play $L$ two periods ahead in time when present aspirations are in $R_I$. By definition, this bound is independent of $\lambda$. For $\alpha \in (2, 3)$, $\hat{p}_{(L,\cdot)}(\alpha)$ is strictly positive because of player 1’s inertia (consider $x_t = (L, \alpha, b, \beta)$). However, even when an action profile $(L,\cdot)$ was played in $t$ there is a strictly positive probability for a 2-step transition away from it, namely first to $(L, N)$ and then to $(H,\cdot)$ (cf. Fig. 11). This implies $\hat{p}_{(L,\cdot)}(\alpha) \leq \hat{p} < 1$ for some $\hat{p} \in (0, 1)$ and all $\alpha \in (2, 3)$.

Similarly, we define for $\alpha \in (2, 3)$

$$\tilde{p}_{(L,N)}(\alpha) \equiv \inf_{\lambda \in (\lambda_4, 1)} \inf_{\beta \in (1, 2)} \min_{(a,b) \in A \times B} \lambda \operatorname{Prob}(a_{t+2}, b_{t+2} = (L, N) \mid x_t = (a, \alpha, b, \beta))$$

to yield a lower bound on the probability that action pair $(L, N)$ will be played in two periods. Obviously, $\tilde{p}_{(L,N)}(\alpha) \leq \hat{p}_{(L,\cdot)}(\alpha)$. Moreover, $\tilde{p}_{(L,N)}(\alpha)$ approaches 0 as $\alpha \downarrow 2$. This is because the probability of player 1 switching away from $H$, $1 - p_1(\alpha - 2)$, converges to 0. Using our upper bounding assumption on players’ inertia functions $p_i$ (cf. Fig. 2), there must, however, exist $\bar{M} > 0$ and $\bar{\alpha} > 0$ with $\hat{p}_{(L,N)}(\alpha) \geq (\alpha - 2)\bar{M}$ for $\alpha \in (2, 2 + \bar{\alpha})$.

For any $\hat{\alpha} \in (0, 1)$, we now choose

$$m(\hat{\alpha}) \equiv \max \left\{ \frac{2 \ln \hat{p}_{(L,N)}(2 + \hat{\alpha})}{\ln \hat{p}_{(L,\cdot)}(2 + \hat{\alpha})}, 3 \right\}.$$  

Since $\hat{p}_{(L,N)}(2 + \hat{\alpha})$ goes to zero but $\hat{p}_{(L,\cdot)}(2 + \hat{\alpha})$ is bounded away from both zero and one, $m(\hat{\alpha}) \to \infty$ as $\hat{\alpha} \to 0$. 

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Finally, we define rectangles $K(\hat{\alpha}) \equiv (2, 2 + \hat{\alpha}) \times (1 + \hat{\alpha} m(\hat{\alpha}), 2)$ for all $\hat{\alpha}$ such that $\hat{\alpha} m(\hat{\alpha}) < 1$, also referred to as permissable $\hat{\alpha}$ (cf. Fig. 12). Using our bounds on $\hat{p}_{(L,N)}(\alpha)$ and $\hat{p}_{(L,Y)}(\alpha)$ for $\alpha \in (2, 3)$, we have
\[
\frac{2 \ln(\hat{\alpha} M)}{\ln \hat{p}} + 3 \geq m(\hat{\alpha}) \geq 3.
\]
From this follows $\hat{\alpha} m(\hat{\alpha}) \to 0$ as $\hat{\alpha} \to 0$, ensuring that there is always an $\hat{\alpha}^*(x)$ independent of $\lambda$ such that all $\hat{\alpha} \in (0, \hat{\alpha}^*)$ are permissable.

**Lemma 5.** From arbitrary state $x_T$ with $(\alpha_T, \beta_T) \in K(\hat{\alpha})$ for an arbitrary permissable $\hat{\alpha}$, a move by unperturbed process $\Phi^0$ into $R_H$ becomes infinitely more likely as $\lambda \to 1$ than a move into $R_N$.

**Proof.** Consider an arbitrary but fixed state $x_T$ with $(\alpha_T, \beta_T) \in K(\hat{\alpha})$ for some fixed permissable $\hat{\alpha}$, and let $\lambda_4$ in the definitions of $\hat{p}_{(L,N)}$ and $\hat{p}_{(L,Y)}$ be large enough such that $\frac{\hat{\alpha}}{2(1-\lambda_4)} \geq 2$. One round of $(L, N)$-play lowers player 1’s aspiration by at least $2(1 - \lambda)$, and player 2’s aspiration by at most the amount $2(1 - \lambda)$. For player 2, the aspiration decrease caused by $(L, N)$-play is greater than that of $(L, Y)$-play.

The probability of a move to $R_H$ is at least as high as that of observing
\[
T^*(\hat{\alpha}, \lambda) \equiv \left[ \frac{\hat{\alpha}}{2(1 - \lambda)} \right]
\]
periods of consecutive \((L, N)\) play ([\(y\)] denoting the smallest integer greater than \(y\)).

\[
T^{**}(\hat{\alpha}, \lambda) \equiv \left[ \frac{\hat{\alpha} m(\hat{\alpha})}{2(1 - \lambda)} \right]
\]

is the minimal number of \((L, \cdot)\)-plays which could decrease player 2’s aspiration level below 1 and hence lead aspirations into \(R_{IV}\).

Starting with \(x_T\) and considering periods \(x_{T+2}, \ldots, x_{T+T^*+2}\), we get \(\hat{p}(L,N)(2 + \hat{\alpha})T^{*(\hat{\alpha},\lambda)}\) as a lower bound on the probability to move into \(R_{II}\).

Similarly, considering periods \(x_{T+2}, x_{T+3}, \ldots\) we get \(\hat{p}(L,\cdot)(2 + \hat{\alpha})T^{**(\hat{\alpha},\lambda)}\) as an upper bound on the probability of moving to \(R_{IV}\).

The ratio of the probabilities of ‘escaping’ from \(K(\hat{\alpha})\) into region \(R_{II}\) and \(R_{IV}\) is

\[
r(\lambda) \equiv \frac{\lambda \text{Prob}\left((\alpha_t, \beta_t) \in R_{II} \text{ for some } t \geq T \mid (\alpha_T, \beta_T) \in K(\hat{\alpha})\right)}{\lambda \text{Prob}\left((\alpha_t, \beta_t) \in R_{IV} \text{ for some } t \geq T \mid (\alpha_T, \beta_T) \in K(\hat{\alpha})\right)}
\]

\[
\geq \frac{\hat{p}(L,N)(2 + \hat{\alpha})^2 \frac{\hat{\alpha} m(\hat{\alpha})}{2(1 - \lambda)}}{\hat{p}(L,\cdot)(2 + \hat{\alpha})^2 \frac{\hat{\alpha} m(\hat{\alpha})}{2(1 - \lambda)}}.
\]

Given the choice of \(m(\hat{\alpha})\), the base term is greater than 1. The exponent goes to infinity as \(\lambda\) approaches 1, implying \(r(\lambda) \xrightarrow{\lambda \to \infty} \infty\).

An analogous result holds for moves from appropriately defined rectangles \(L(\hat{\beta})\) (see Fig. 12) into \(R_{IV}\) and \(R_{II}\), respectively.

**Proof of Theorem 2.** Let \(s_{jk}\) refer to elements of the matrix defined in the proof of Theorem 1, where \(s_{LY, LN}\) and \(s_{H, LN}\) approach 0 for \(\lambda \to 1\).

Our assumptions on \(g_1\) and \(g_2\) ensure that a positive measure of perturbations from \(c_{LY}\) or \(c_{H}\) stays in a neighborhood comprised in \(L(\hat{\beta})\) or \(K(\hat{\alpha})\) respectively for permissible \(\hat{\beta}\) and \(\hat{\alpha}\), implying an almost sure return to \(c_{LY}\) or \(c_{H}\), respectively, as \(\lambda \to 1\). So, \(s_{LY, LY}, s_{H, H} \geq \nu > 0\).
For case i), choose the supports of \( g_i(\cdot | c_H) \) \((i = 1, 2)\) wide enough to place positive weight on some rectangle \( L(\hat{\beta}) \) for a permissable \( \hat{\beta} \), but choose \( g_i(\cdot | c_{LY}) \) such that they place all weight on some rectangle \( L(\hat{\beta}') \) for a permissable \( \hat{\beta}' \) and \( R_{IV} \). Then, \( s_{LY,H} < \varepsilon(\lambda) \) but \( s_{H\cdot,LY} \geq \delta > 0 \). As \( \lambda \to 1, \varepsilon(\lambda) \to 0 \) by Lemma 2 and the analogue of Lemma 5. In the limit \( c_{LY} \) is the only absorbing state of \( \Phi^\eta \), and \( \mu^* \) places all weight on it. Case ii) is analogous. For case iii) choose the supports of \( g_i(\cdot | c_H) \) \((i = 1, 2)\) wide enough to place positive weight on some rectangle \( L(\hat{\beta}) \) for a permissable \( \hat{\beta} \), and choose those of \( g_i(\cdot | c_{LY}) \) wide enough to place positive weight on some rectangle \( K(\hat{\alpha}) \) for a permissable \( \hat{\alpha} \). Now, both \( s_{LY,H} \) and \( s_{H\cdot,LY} \) are bounded away from zero. Consequently, \( \mu^* \) will place positive weight on both \( c_{LY} \) and \( c_H \) even in the limit \( \lambda \to 1 \).

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