Inferior Players in Simple Games[†]

Stefan Napel[*] and Mika Widgrén[**]

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Abstract

Power indices like those of Shapley and Shubik (1954) or Banzhaf (1965) measure the distribution of power in simple games. This paper points at a deficiency shared by all established indices: players who are inferior in the sense of having to accept (almost) no share of the spoils in return for being part of a winning coalition are assigned substantial amounts of power. A strengthened version of the dummy axiom based on a formalized notion of inferior players is a possible remedy. The axiom is illustrated first in a deterministic and then a probabilistic setting. With three axioms from the Banzhaf index, it uniquely characterizes the Strict Power Index (SPI). The SPI is shown to be a special instance of a more general family of power indices based on the inferior player axiom.

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[*]WIOR, Universität Karlsruhe, D-76128 Karlsruhe, Germany. Tel. +49-721-608-2697 and -3082 (fax); e-mail: napel@wior.uni-karlsruhe.de

[**]Turku School of Economics and Centre for Economic Policy Research (CEPR), Rehtorinpellonkatu 3, 20500 Turku, Finland. Tel. +358-2-3383 337 and 3383 302 (fax); e-mail: mika.widgren@tukkk.fi

1 Introduction

Power indices are functions that map n-person simple games, such as weighted multi-party voting games, to n-dimensional real vectors. They assign to each player a number that indicates the player's power to shape events. The power of a dictator is usually scaled to unity and that of null or dummy players, who cannot turn any losing coalition into a winning one, is set to zero. Anonymity, particular monotonicity and aggregation properties for different simple games are additionally required or, at least, desired.

Power indices have been applied to evaluate numerous political and economic institutions. Power distributions in the context of shareholders' meetings have been one focus of attention (compare e. g. Leech 1988), with the related theoretical challenge of dealing with cross-ownership whereby players exert power both directly and indirectly (see Gambarelli and Owen 1994 for one solution). In the political sphere, decision making in the U.S. Congress, U.S. presidential elections (see Owen 1975), the U.N. Security Council, and, recently, the institutions of the European Union (e. g. Laruelle and Widgrén 1998; see Nurmi 1998 for a comprehensive survey) have all been studied extensively using power indices.

Despite the wide application and almost fifty years after the seminal contribution to power measurement by Shapley and Shubik (1954), there is still considerable controversy as to what constitutes an appropriate power measure.[1] In the wake of Shapley and Shubik's work, numerous power indices have been proposed and axiomatically characterized – most notably by Banzhaf (1965), Deegan and Packel (1978), and Holler and Packel (1983).[2]

[1] Cf. the contributions in Holler and Owen (2001), for example.

[2] For a recent comparative investigation of power indices, their properties and applicability, see Felsenthal and Machover (1998).

However, none of these indices is consistent with traditional notions of competitive equilibrium or the cooperative concept of the core: in a three-player simple game where the only winning coalitions are the grand coalition ABC and the two coalitions AB and AC, core and competitive analysis attribute *all* power to player A. In contrast, the indices of Shapley-Shubik, Banzhaf, Deegan-Packel, or Holler-Packel respectively assign $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{2}$, and $\frac{1}{2}$ of total power to players B and C.[3]

In this paper, we define the concept of *inferior players* as a first step to overcome this deficiency. Based on this definition we suggest to replace the dummy axiom conventionally used in power measurement by a stricter axiom. The proposed axiom requires indices to *not* take into account a player's supposed power (as traditionally measured by swings, pivot positions, etc.) if some other player can issue the following ultimatum to him: accept (almost) no share of the spoils from a winning coalition or be prevented from taking part in one at all. Thus, power measurement is brought more in line with competitive analysis.

Section 2 starts with some preliminary definitions. Section 3 introduces the concept of inferior players and proposes the inferior player axiom. The Strict Power Index (SPI), related to the Banzhaf index, is introduced and axiomatized in section 4. Then, section 5 investigates inferiority in the realm of probabilistic power measurement. A probabilistic foundation of the SPI and a more general family of indices is given, before section 6 concludes.

[3]Note that successful attempts have been made to provide a non-cooperative foundation for the value concepts related to power indices, most notably the Shapley value (see Hart and Mas-Collel, 1996, for example). Doubts about the realism of the highly specific bargaining procedures and respective limit considerations are, in our view, confirmed by this simple example.

2 Preliminary definitions

Let $N = \{1, 2, \ldots, n\}$ be the set of players. $\mathcal{P}(N) = \{0,1\}^n$ is the set of feasible coalitions. The *simple game* v is characterized by the set $W(v) \subsetneq \mathcal{P}(N)$ of *winning coalitions*. $W(v)$ satisfies $\varnothing \notin W(v)$, $N \in W(v)$ and $S \in W(v) \wedge S \subseteq T \Rightarrow T \in W(v)$. v can also be described by a characteristic function $v : \mathcal{P}(N) \to \{0,1\}$ with

$$v(S) = \begin{cases} 0 & \text{iff } S \notin W(v), \\ 1 & \text{iff } S \in W(v). \end{cases}$$

$\mathcal{G}^N$ denotes the set of all such n-person simple games. *Voting games* are special instances of simple games that are characterized by a non-negative real vector $r_v = (q; w_1, \ldots, w_n)$, where w_i represents player i's voting weight and q represents the quota of votes that establishes a winning coalition.

A player who by leaving a winning coalition $S \in W(v)$ turns it into a losing coalition $S \setminus \{i\} \notin W(v)$ has a *swing* in S, and is called a *crucial member* of coalition S. Coalitions where player i has a swing are called *crucial coalitions with respect to* i. Let

$$C_i(v) := \{S \subseteq N \mid S \in W(v) \ \wedge \ S \setminus \{i\} \notin W(v)\}$$

denote the set of crucial coalitions w.r.t. i. A concise description of v is given by

$$M(v) := \{S \subseteq N \mid S \in W(v) \ \wedge \ \forall i \in S : \ S \setminus \{i\} \notin W(v)\},$$

the set of *minimal winning coalitions (MWC)*. The number of swings of player i will be denoted by

$$\eta_i(v) := \mid C_i(v) \mid .$$

A player i with $\eta_i(v) = 0$ is called a *dummy player*.

A power index is a mapping $\mu : \mathcal{G}^N \to \mathbb{R}^n_+$, assigning to each player $i \in N$ a number $\mu_i(v)$ that indicates i's power in the considered game v. Typically, one scales μ such that $\mu_i(v) = 1$ if and only if i is a dictator in v, i. e. $M(v) = \{\{i\}\}$. Moreover, it is required that $\mu_i(v) = 0$ if i is a dummy player. A prominent example, the *non-normalized Banzhaf index (BZI)* β, is defined by

$$\beta_i(v) := \frac{\eta_i(v)}{2^{n-1}}, \qquad i \in N.$$

Since there are 2^{n-1} coalitions in which i could have a swing, $\beta_i(v)$ represents i's ratio of actual to potential number of swings.

An index μ is *anonymous* if $\mu_{\pi(i)}(\pi v) = \mu_i(v)$ holds for any permutation π of the set N of players, where πv is defined by $(\pi v)(S) := v(\pi^{-1}(S))$.[4] μ is *locally monotonic* on the domain of voting games if $w_i \geq w_j$ in r_v implies $\mu_i(v) \geq \mu_j(v)$, i. e. more weight implies more power. The Shapley-Shubik index (SSI), BZI, and normalized BZI are locally monotonic, the Deegan-Packel or Holler-Packel indices are not.

Monotonicity is also defined with respect to players' positions in different simple games (cf. e. g. Levínský and Silárszky 2001). $u \in \mathcal{G}^N$ can be considered 'better' than $v \in \mathcal{G}^N$ from player i's point of view if all winning coalitions of v with i also win in u (and, possibly, some other coalitions with i win in u) and if all winning coalitions of u without i also win in v (and possibly some more). Formally, define $\succ_i$ with

$$u \succ_i v \;:\Longleftrightarrow\; \begin{cases} \quad i \in S \wedge S \in W(v) \;\Rightarrow S \in W(u) \\ \wedge \;\; i \notin S \wedge S \in W(u) \;\Rightarrow S \in W(v). \end{cases}$$

An index μ is *globally monotonic* if $u \succ_i v$ implies $\mu_i(u) \geq \mu_i(v)$ for all $i \in N$. Provided that an index is anonymous, global monotonicity implies

[4] *Weighted values* serve as an example where anonymity is relaxed (see Kalai and Samet 1988 for details). A recent related study is Haimanko (2000).

local monotonicity (Levínský and Silárszky 2001). SSI and BZI are globally monotonic, the normalized BZI is not.

Power in simple games can also be analysed in a probabilistic setting. Instead of deterministic coalitions $S \subseteq N$, corresponding to corner points $s \in \{0,1\}^n$ of the n-dimensional unit cube, one considers fuzzy or random coalitions $\mathfrak{S}$ represented by points $p \in [0,1]^n$ anywhere in the cube. Each $p_i \in [0,1]$ is interpreted as the probability of player $i \in N$ deciding in favour of a random proposal or of participating in a randomly formed coalition. We refer to p_i as player i's *acceptance rate*.

Players' acceptance decisions are assumed to be independent. Thus, the probability of forming a given coalition $S \subseteq N$ is $\Pr(\mathfrak{S} = S) = \Pi_{i \in S} p_i \Pi_{j \notin S}(1 - p_j)$. The characteristic function $v : \{0,1\}^n \to \{0,1\}$ of a simple game can then be extended by weighting $v(S)$ for all coalitions $S \subseteq N$ with their respective probability of formation. We get the *multilinear extension (MLE)* $f : [0,1]^n \to [0,1]$ of game v (see Owen 1972):

$$\begin{aligned} f(p_1, \ldots, p_n) &= \sum_{S \subseteq N} \prod_{i \in S} p_i \prod_{j \notin S} (1 - p_j)\, v(S) \\ &= \sum_{S \in W(v)} \prod_{i \in S} p_i \prod_{j \notin S} (1 - p_j)\,. \end{aligned}$$

For fixed acceptance rates $(p_1, \ldots, p_n)$, the MLE gives the probability of formation of a winning coalition in v, and also the expected value of v. Note that players' acceptance rates may not be constants, but random variables themselves.

Let f_i denote the partial derivative $\partial f / \partial p_i$ of v's MLE with respect to p_i. It is usually referred to as player i's *power polynomial* (Straffin 1988). $f_i(p_1, \ldots, p_n)$ is the probability of i having a swing in the random coalition to be formed in game v. When players' acceptance rates $(p_1, \ldots, p_n)$ are themselves random variables with a joint distribution P, the expectation

$$Ef_i = \int f_i\left(p_1, \ldots, p_n\right) dP \tag{1}$$

is an indicator of i's power in game v. The *probabilistic power index* defined by (1) coincides with traditional deterministic indices for several plausible probability models. When all players' acceptance rates are independently drawn from a uniform distribution on $[0,1]$ – in short notation: $\forall i \in N: p_i \overset{i.i.d.}{\sim} U[0,1]$ – then (1) equals the BZI.[5]

3 Inferior Players

In the introduction, the game v_1 with $W(v_1) = \{AB, AC, ABC\}$ illustrated the divergence between power predictions based on conventional indices on the one hand, and competitive analysis or the core on the other hand. In the considered game, player A can credibly issue an ultimatum to B (or C) in which A proposes to establish coalition AB in return for (in the limit) total concession by B on those economic or policy issues related to the formation of a winning coalition on which A and B have opposing interests. Thus, B is robbed of the power commonly associated with his swing. The possibility of A flipping a coin before B or C is chosen to establish a winning coalition extends the argument to both players.

Describing B's position a bit more abstractly, it can be said that there exists a player who can veto all coalitions in which B makes a positive contribution, i. e. is crucial, but who can herself form a crucial coalition without an opportunity for B to interfere. Threatened by A taking this outside option, B prefers (almost) any concession to A's demands to being excluded from

[5]The SSI can be derived from the more restricting assumption where $\forall i \in N: p_i = t$ and t is uniformly distributed on $[0,1]$.

winning. In this sense, B is an *inferior player* in game v. Formalizing this intuitive notion of inferiority, we state:

Definition 1: Player i is *inferior* in simple game v if $\exists j \neq i$:

$$\begin{aligned} \forall S \in C_i(v): &\quad j \in S \\ \wedge \quad \exists S' \in C_j(v): &\quad i \notin S' \end{aligned}$$

Let $I(v) \subsetneq N$ denote the set of inferior players in v. There is a neat equivalent definition:

Proposition 1: Player i is inferior in $v \in \mathcal{G}^N \iff \exists j \neq i: \; C_i(v) \subsetneq C_j(v)$.

Proof: a) Let i be inferior in v. Assume that there exists $\tilde{S} \in C_i(v)$ with $\tilde{S} \notin C_j(v)$. It follows that $\tilde{S} \in W(v)$, and $\tilde{S} \backslash \{j\} \in W(v)$. Furthermore, from $\tilde{S} \backslash \{i\} \notin W(v)$ it follows that $\tilde{S} \backslash \{j\} \backslash \{i\} \notin W(v)$. Thus, $\tilde{S} \backslash \{j\} \in C_i(v)$ – a contradiction to $\forall S \in C_i(v): j \in S$. So $C_i(v) \subseteq C_j(v)$. Because j is crucial in at least one coalition S' without i, we have $C_i(v) \subsetneq C_j(v)$.

b) $S \in C_j(v)$ implies $j \in S$ – establishing the first part of definition 1. Assume $C_i(v) \subsetneq C_j(v)$ and $\forall S' \in C_j(v): i \in S'$. Using the argument in a), the latter implies $C_j(v) \subseteq C_i(v)$. This is a contradiction. □

Any dummy player is inferior. The reverse is true for *strong* or *decisive simple games* where $\forall S \subseteq N: \; S \in W(v) \; \vee \; N \setminus S \in W(v)$. A player i can be inferior because of a player j who is himself inferior. However, by proposition 1 and the transitivity of $\subsetneq$, there is at least one non-inferior player k who makes i inferior. Existence of inferior players does not require a veto player such as A in v_1. If player i is inferior in v, $x_i = 0$ for any element x of v's core. Rotating members of the UN Security Council are a real-world example of inferior players.

Players who are not inferior are generally agreed to be powerful. The conventional notion of powerless players embodied in the *dummy player axiom* – requiring that an index μ gives zero power to dummy players – is a quite weak one, however. In our view, it is too weak for a relevant class of circumstances that are modelled by simple games – in particular, if there is scope for negotiation before coalition formation and there are at most finitely many decisions to be taken. Under these circumstances, an inferior player i is subject to aforementioned credible ultimatum threats by some player j. The power usually associated with the swings that an inferior player may have is obliterated, and an inferior player can be expected to have only marginal influence on any economic or political decision. Therefore, we suggest to strengthen the conventional dummy player axiom:

Inferior Player Axiom: i is inferior in $v \Longrightarrow \mu_i(v) = 0$.

As illustrated in the introduction, none of the conventional power indices satisfies the inferior player axiom.

4 The Strict Power Index (SPI)

In order to show that the inferior player axiom leads to reasonable power indices with desirable properties and plausible probability models, we will define an example index related to the BZI. This is based on the traditional deterministic formulation of power indices. Note that similar adaptations could be made to the SSI, the Deegan-Packel index, or other power indices. We start with the following adaptation of the notion of swings:

Definition 2: Player i has a *strict swing* in winning coalition $S \in W(v)$ if

a) i can turn S into a losing coalition by leaving it, and

b) i is not inferior in v, i. e. $i \notin I(v)$.

Let

$$\tilde{\eta}_i(v) := \begin{cases} \mid C_i(v) \mid & \text{iff } i \notin I(v), \\ 0 & \text{iff } i \in I(v) \end{cases}$$

denote the number of strict swings of player i in game v. Substituting strict swings for swings in the definition of the BZI, we get the following new power index:

Definition 3: The *Strict Power Index (SPI)* $\sigma : \mathcal{G}^N \to \mathbb{R}^n_+$ is given by

$$\sigma_i(v) := \frac{\tilde{\eta}_i(v)}{2^{n-1}}, \qquad i \in N.$$

By construction, $\sigma_i(v) = 0$ if and only if player i is inferior, and $\sigma_i(v) = 1$ if and only if i is a dictator. For the example game v_1, the SPI produces the vector $\sigma(v_1) = (\frac{3}{4}, 0, 0)$; A is the only powerful player in v_1, but still no dictator. The game v_2 with $N = \{A, B, C, D, E, F\}$ and $M(v_2) = \{ABC, ABD, ACE, BDEF\}$ illustrates that SPI and BZI index can imply different power rankings: $\sigma(v_2) = (\frac{7}{16}, \frac{5}{16}, 0, 0, \frac{3}{16}, 0)$ and $\beta(v_2) = (\frac{7}{16}, \frac{5}{16}, \frac{4}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{16})$. C is part of smaller MWC than E. This yields a greater number of swings so that greater power is indicated by the BZI. However, C's supposed power is obliterated by his dependence on A. So, E has more strict swings that actually translate into power. Corresponding with the BZI, we have the following result:

Proposition 2: The SPI is globally monotonic.

Proof: Consider arbitrary simple games $u, v \in \mathcal{G}^N$ with $u \succ_i v$. We need to show that $\sigma_i(u) \geq \sigma_i(v)$. If i is inferior in v, this is trivial. The global monotonicity of the BZI implies $\sigma_i(u) \geq \sigma_i(v)$ if i is not inferior in u. It remains to confirm that i cannot be inferior in u without being inferior in v.

It can be checked that $u \succ_i v$ implies $C_i(v) \subseteq C_i(u)$. Now suppose that i is *not* inferior in v. For any player $j \neq i$, either $\exists S_j \in C_i(v): \; j \notin S_j$, but then $S_j \in C_i(u)$ with $j \notin S_j$. Or $C_i(v) = C_j(v)$. Player i keeps his swings in all coalitions $S \in C_i(v)$ in game u. If either j has additional swings in u only together with i, or if there is a new coalition $S \in C_i(u)$ with $j \notin S$, we are finished. Otherwise, for i to become inferior in u, it must be true that a) j is part of all $S \in C_i(u)$ and that b) there is a coalition $\hat{S} \in C_j(u)$ with $i \notin \hat{S}$. $u \succ_i v$ implies $\hat{S} \in W(v)$. Now, we either have $\hat{S} \in C_j(v)$, which contradicts $C_i(v) = C_j(v)$. Or $\hat{S} \notin C_j(v)$, i.e. $\hat{S} \setminus \{j\} \in W(v)$. Since $\hat{S} \setminus \{j\} \cup \{i\}$ wins in v, it also wins in u. Player i cannot be crucial in $\hat{S} \setminus \{j\} \cup \{i\}$ because that would contradict a). So, $\hat{S} \setminus \{j\} \in W(u)$, contradicting b). $\square$

It facilitates comparisons with other power measures if an index is fully characterized by a set of logically independent axioms. We provide an axiomatic characterization of the SPI along the same route which Dubey and Shapley (1979) take to axiomatize the Banzhaf index. For this, let the simple game $u \vee v$ be defined by the characteristic function $(u \vee v)(S) := \max\{u(S), v(S)\}$ for all $S \subseteq N$. Similarly, define $u \wedge v$ by $(u \wedge v)(S) := \min\{u(S), v(S)\}$. Any simple game $v \in \mathcal{G}^N$ can be defined as the composition $u_{S_1} \vee \ldots \vee u_{S_r}$, where $S_1, \ldots, S_r \in M(v)$ are the MWC in v and where u_{S_k} is the auxiliary game in which exactly all coalitions containing S_k are winning.

Some power measures, most notably BZI and SSI, are based on a linear notion of power. This explicitly requires from a power index μ that the *additivity axiom* holds: $\forall u, v \in \mathcal{G}^N : \mu(u \vee v) = \mu(u) + \mu(v) - \mu(u \wedge v)$. Consider e. g. the players $N = \{A, B, C, D\}$, and games $v_3, v_4 \in \mathcal{G}^N$ with $M(v_3) = \{AB, AC\}$ and $M(v_4) = \{AD, BCD\}$. According to the BZI, B's power in $v_3 \vee v_4$ is simply the sum of power in v_3 and v_4, $\frac{1}{4} + \frac{1}{8}$, corrected by $-\frac{1}{8}$ for i's swing ABD from v_3 that becomes void due to overlap with v_4. This does not hold for the SPI: B is inferior in v_3 and v_4, but not $v_3 \vee v_4$. $\sigma_B(v_3) = \sigma_B(v_4) = 0$ is contrasted by $\sigma(v_3 \vee v_4) = (\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, i. e. in the composed game, B is even as powerful as player D who made B inferior in v_4. The strategic considerations underlying the inferior player axiom imply that power is generally non-linear. We therefore use a less restrictive requirement than additivity for the characterization of the SPI.[6]

Proposition 3: The SPI is the unique power index satisfying the following four independent axioms:

A1: (inferior players) i is inferior in $v \Longrightarrow \mu_i(v) = 0$.

A2: (absolute power) $\sum_{i=1}^{n} \mu_i(v) = \frac{1}{2^{n-1}} \sum_{i=1}^{n} \tilde{\eta}_i(v)$.

A3: (anonymity) For any permutation π of N: $\mu_{\pi(i)}(\pi v) = \mu_i(v)$.

A4: (aggregation) i is not inferior in v
$\Longrightarrow \mu_i(v) \equiv \mu_i(\bigvee_{S \in M(v)} v_S) = \sum_{T \subseteq (M(v))} (-1)^{|T|-1} \mu_i(\bigwedge_{S \in T} v_S)$.

Proof: A1 and A2 are satisfied by construction. A3 follows from the anonymity of swings, and hence of strict swings. A4 refers to non-inferior players only. For those players, the SPI is constructed to coincide with the BZI. By complete induction, we prove a lemma claiming that A4 is satisfied by any index μ which satisfies additivity, in particular the BZI: Consider an arbitrary game $w^r \in \mathcal{G}^N$ with exactly $r \geq 1$ MWC, i. e. $M(w^r) = \{S_1, \ldots, S_r\}$. The claim is obviously true for $r = 1$. We proceed to $r + 1$ and consider $w^{r+1} \in \mathcal{G}^N$ with $M(w^r) = \{S_1, \ldots, S_r, S_{r+1}\}$. Using additivity and the result

[6] Our aggregation axiom can, in fact, replace additivity in the axiomatization of BZI or SSI if its restriction to non-inferior players is dropped.

for r, $\mu_i(w^{r+1})$ equals

$$\mu_i(w^r \vee u_{S_{r+1}}) = \Big[\sum_{T\subseteq\mathcal{P}(\{S_1,\ldots,S_r\})} (-1)^{|T|-1}\mu_i(\bigwedge_{S\in T} u_S)\Big] + \mu_i(u_{S_{r+1}}) - \mu_i(w^r \wedge u_{S_{r+1}}). \quad (2)$$

$\mu_i(w^r \wedge u_{S_{r+1}})$ is equivalent to $\mu_i\Big(\bigvee_{S\in M(w^r)} (u_S \wedge u_{S_{r+1}})\Big)$. To this, the result for r can be applied once more:

$$\begin{aligned}
\mu_i(w^r \wedge u_{S_{r+1}}) &= \sum_{T\subseteq\mathcal{P}(\{S_1,\ldots,S_r\})} (-1)^{|T|-1}\,\mu_i\Big(\bigwedge_{S\in T} (u_S \wedge u_{S_{r+1}})\Big) \\
&= -\sum_{T\subseteq\mathcal{P}(\{S_1,\ldots,S_r\})} (-1)^{|T\cup\{S_{r+1}\}|-1}\,\mu_i(\bigwedge_{S\in T\cup\{S_{r+1}\}} u_S).
\end{aligned}$$

Substituting this in (2) proves the claim for $r+1$, and thus the lemma.

Next, we prove that A1–A4 uniquely define a function $\mu : \mathcal{G}^N \to \mathbb{R}^n_+$. We first consider games with a single minimal winning coalition $S \subseteq N$, i. e. the auxiliary game u_S. All players $i \notin S$ are inferior in u_S and hence by A1 $\mu_i(u_S) = 0$. All non-inferior players $j \in S$ by A3 have the same power $\mu_j(u_S) = a$ with $a \geq 0$. Thus, we have $\sum_{i=1}^n \mu_i(u_S) = a|S|$. A2 requires $a|S| = \frac{1}{2^{n-1}} \sum_{i=1}^n \tilde{\eta}_i(u_S)$. By construction of u_S we have

$$\tilde{\eta}_i(u_S) = \begin{cases} 0 & \text{iff } i \notin S, \\ 2^{n-|S|} & \text{iff } i \in S, \end{cases}$$

implying

$$a = \frac{1}{2^{|S|-1}}.$$

Thus, μ is uniquely defined for all auxiliary games u_S with $S \subseteq N$. A1 and A4 extend this definition to the entire domain $\mathcal{G}^N$.

Finally, independence of A1–A4 need to be demonstrated. The BZI β obviously violates A1, but obeys A2–A4. The normalized version of the SPI, $\sigma(v)/\sum_i \sigma_i(v)$, violates A2 but obeys the remaining axioms. An index consistent with A1, A2, and A4, but not A3 is obtained by allocating the number of strict swings in single-MWC auxiliary games to the non-inferior player with lowest order number, using A1 and A4 to extend this to non-auxiliary games with multiple MWC. Indices satisfying A1–A3 that violate A4 will be given in proposition 5 (using $c \neq \frac{1}{2}$). □

5 Inferior players in a probabilistic setting

In the probabilistic setting, the property of $i \in I(v)$ being an inferior player has to be reflected in some way by i's acceptance rate p_i. We can find a plausible restriction on p_i by recalling that inferior players have to content themselves with essentially a zero share of economic or political spoils when belonging to a winning coalition. This means that an inferior player is basically indifferent between joining a winning coalition or staying outside, between voting for or against a proposal. This can be formalized by:

Strict Power Condition (SPC): i is inferior in $v \Longrightarrow p_i \equiv \frac{1}{2}$.

One gets the following probabilistic foundation of the SPI:

Proposition 4: Applying the SPC in the setting of the probabilistic BZI, i. e.

$$p_i \begin{cases} \equiv \frac{1}{2} & \text{iff } i \in I(v), \\ \overset{i.i.d.}{\sim} U[0,1] & \text{iff } i \notin I(v), \end{cases}$$

implies the probabilistic SPI.

The proposition follows from the more general proposition 5 below ($c = \frac{1}{2}$). Note that imposition of the SPC changes the interpretation of power polynomial $f_i(p_1, \ldots, p_n)$. It no longer gives the probability of player i having a swing or being crucial in the random coalition that is to be formed, but the probability of player i having a strict swing or of being crucial in a way that actually permits exertion of power.

Inferior players' practical indifference towards being part of a winning coalition can, of course, be formalized differently. For example, one could assume that inferior players join whatever coalition is decided on by the powerful players of the game with probability one, or probability zero, or some probability c in between. This leads to the

Generalized Strict Power Condition (GSPC): i is inferior in $v \Longrightarrow p_i \equiv c,\ \ c \in [0,1]$.

The GSPC restricts the domain of v's MLE to the $(n-m)$-dimensional unit cube, where $m :=\mid I(v) \mid$ denotes the number of inferior players in v. In order to characterize those deterministic indices whose probabilistic counterpart satisfies the GSPC for some $c \in [0,1]$ we need to decompose $\tilde{\eta}_i(v)$ and generalize the notion of strict swings.

Definition 4: Player i has a θ*-swing* in winning coalition $S \subseteq W(v)$ if

a) i can turn S into a losing coalition by leaving it,

b) i is not inferior in v, i. e. $i \notin I(v)$, and

c) exactly θ inferior players are part of S.

Let

$$\eta_i^{(\theta)}(v) :=\mid \{S \subseteq N \ \mid \ S \in C_i(v) \ \wedge \ i \notin I(v) \ \wedge \ |S \cap I(v)| = \theta\} \mid$$

denote the number of θ-swings of player i in game v. We trivially have

$$\sum_{\theta=0}^{m} \eta_i^{(\theta)}(v) = \tilde{\eta}_i(v).$$

Various indices can be defined based on the primitive θ-swing. Averaging $\eta_i^{(\theta)}(v)$ with particular weights on each θ may incorporate especially plausible or empirically relevant assumptions about inferior players' behaviour. A continuum of anonymous power indices which satisfy the inferior player axiom can now be probabilistically characterized:[7]

Proposition 5: A MLE satisfying the GSPC gives zero power for inferior players. Applying the GSPC in the setting of the probabilistic BZI, i. e.

$$p_i \begin{cases} \equiv c & \text{iff } i \in I(v), \\ \overset{i.i.d.}{\sim} U[0,1] & \text{iff } i \notin I(v) \end{cases}$$

for some $c \in [0,1]$, implies the *Generalized Strict Power Index (GSPI)* σ^c with

$$\sigma_j^c(v) := \sum_{\theta=0}^{m} c^\theta \,(1-c)^{m-\theta} \,\frac{\eta_j^{(\theta)}(v)}{2^{n-m-1}}$$

Proof: W.l.o.g. consider $v \in \mathcal{G}^N$ with non-inferior players $1, \ldots, n-m$. If v's MLE $f(p_1, \ldots, p_n)$ satisfies the GSPC, then it is a non-degenerate function only of $p_1, \ldots, p_{n-m}$. But clearly $\partial f(p_1, \ldots, p_{n-m})/\partial p_i = 0$ for $i > n-m$.

Imposing the GSPC to a MLE gives

$$\begin{aligned} f(p_1, \ldots, p_n) &= \sum_{S \in W(v)} \prod_{\substack{i \in I(v) \\ i \in S}} c \prod_{\substack{j \in I(v) \\ j \notin S}} (1-c) \prod_{k \in S \setminus I(v)} p_k \prod_{l \notin S \cup I(v)} (1-p_l) \\ &= \sum_{S \in W(v)} c^{\theta(S)} \,(1-c)^{m-\theta(S)} \prod_{k \in S \setminus I(v)} p_k \prod_{l \notin S \cup I(v)} (1-p_l), \end{aligned}$$

where $\theta(S)$ indicates the number of inferior players in coalition S. Taking expectations of the partial derivative with respect to p_i for $i \notin I(v)$ yields

$$Ef_i(p_1, \ldots, p_n) = \sum_{S \in C_i(v)} c^{\theta(S)}(1-c)^{m-\theta(S)} \left(\frac{1}{2}\right)^{n-m-1}.$$

Considering only one $S \in C_i(v)$ for each number $\theta \in \{0, \ldots, m\}$ of inferior members and weighting this summand by the number $\eta_i^{(\theta)}(v)$ of such coalitions in $C_i(v)$ produces the claim. □

[7] Absolute power axiom A2 is satisfied by appropriate re-scalings.

The SPI is the special case of $c = \frac{1}{2}$. The special case of $c = 1$ corresponds to the Follower-Leader Index of Power (FLIP) defined in Napel and Widgrén (2000).[8] The FLIP is suited to environments in which inferior players are so benign that they always follow the leaders of the game into whatever coalition the latter want to establish.

6 Conclusion

In this paper we argue in favour of strengthening the commonly used dummy player axiom of power measurement to an axiom based on the concept of inferior players. Motivation for this is the discrepancy between power indications given by, on the one hand, established indices that are based on the dummy player axiom and, on the other hand, an important aspect of power related to competitive equilibrium and core analysis.

In order to demonstrate that meaningful indices which comply with the inferior player axiom can be constructed, we proposed the Strict Power Index (SPI). It was first analyzed in a traditional deterministic setting and axiomatized. For a comprehensive understanding of the concept of inferior players we then investigated its probabilistic counterpart; a probabilistic condition that implies the SPI was derived, and generalized.

Future research may apply the inferior player axiom to other indices than the non-normalized Banzhaf index, e. g. those of Shapley and Shubik or of Deegan and Packel. It could be worthwhile to investigate more thoroughly the mathematical properties of the respective adaptations of the Banzhaf,

[8] The requirement $p_i \equiv 1$ for $i \in I(v)$ can be weakened to the restriction that $p_i p_j = p_j$ if i is inferior to j in v. This asks for a stronger type of behavioural similarity than the full correlation assumption of the SSI, where inferior players follow the common standard t rather than other players j.

Shapley-Shubik or Deegan-Packel index in terms of axiomatization, monotonicity, and susceptibility to typical paradoxes in power measurement. The inferior player axiom could be extended to the domain of general games in characteristic function form. One may also define a stability concept for coalition structures, i. e. partitions of the set of players, requiring that no element is inferior in the reduced game among coalitions. For example, the stable structures in v_1 are $\{A, BC\}$ and $\{ABC\}$. The relations to the traditional stability notions are yet unexplored.

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