Voting Power*

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This version: January 26, 2017

Abstract

The chapter investigates ways to quantify how institutional rules, such as different voting weight arrangements in councils or two-tier voting systems, allocate influence on outcomes among the collective decision makers. After basic concepts of the common binary voting framework have been laid out, several indices of voting power are introduced with their key properties and probabilistic interpretations. Techniques for the computation of these indices, limit results for large voting bodies and the problem of designing rules with desirable power implications are discussed. Possibilities to analyze the distribution of influence in richer than just binary voting scenarios are pointed out.

Keywords: simple voting game, weighted voting game, influence measure, power index, Shapley-Shubik index, Penrose-Banzhaf index, Deegan-Packel index, Holler-Packel index, public good index, nucleolus, axiomatic characterization, enumeration of coalitions, generating function method, Monte Carlo approximation, Penrose limit theorem, square root rule, linear rule, inverse problem, non-binary voting, strategic voting

Prepared for:

Oxford Handbook of Public Choice, edited by R. Congleton, B. Grofman and S. Voigt, Oxford University Press, forthcoming.

*The author is grateful to Bernard Grofman, Sascha Kurz, Nicola Maaser, Alexander Mayer, Hannu Nurmi and Marina Uzunowa for helpful comments on earlier drafts. The usual caveat applies.

1 Introduction

Suppose that three decision makers respectively command 48%, 37% and 15% of votes in a council. They might be delegates from three differently sized constituencies, represent three perfectly disciplined parties in a parliament, or they could be stockowners with respective shareholdings in a private company. Proposals need to be supported by a simple majority of votes in order to be passed.

What advantage does commanding the large voting share of 48% give to the first decision maker, compared to his peer with only 15% of total votes? More generally, how do asymmetric roles of voters under a given decision rule allocate the ability to influence the collective decision? This ability is commonly referred to as the *voting power* of members of a collective decision making body. Prominent targets of corresponding power investigations have been the US Electoral College, the Council of the EU, the UN Security Council, or the Board of Governors of the IMF.

The analysis usually adopts an *a priori* perspective. This means that investigations of voting power are not concerned with a specific issue, with a corresponding realization of preferences, alliances, and factual arguments. They rather try to assess average or expected asymmetries in influence on outcomes in general. These asymmetries may be caused by non-uniform voting weights but also other institutional arrangements like agenda rights, veto powers, etc.

When an actual decision is taken, effects of the voting rule interact with – and may *a posteriori* be dominated by – the circumstantial alignment of interests among the decision makers, their strategic and rhetorical skills, personal friendships, and so forth. Hence, knowing the distribution of (a priori) voting power may help little for predicting a specific voting outcome. It is instructive nonetheless to assess voting power for institutional reasons: in order to compare the roles assigned to different agents, to identify winners and losers from a rule change, or in order to design voting rules so that they generate – at least from behind a constitutional 'veil of ignorance' – a level playing field for the interests involved.

Assessing voting power may look trivial at first sight. But, on closer inspection, it turns out to be misleading that the first player's weight in our example (48%) is more than triple that of the third decision maker (15%): given that only a simple majority of votes is needed in order to pass proposals, *any* coalition of two of the voters is sufficient. So *any* of the three council members needs just one ally, no matter if its vote share or voting weight is 48%, 37% or 15%. Therefore, all three can be expected to wield identical voting power under the decision rule at hand.

As in the example, voting power is typically not proportional to voting weight. Moreover, many voting rules cannot even be expressed by a simple list of voting weights – for instance, when legislation requires approval by two parliamentary chambers with a veto option for the president. One may want to evaluate who is given how much leverage over outcomes also for such decision arrangements.

An *index of voting power* is an analytical tool that is designed for the purpose of such evaluation. A great multitude of distinct indices have been proposed, all trying to quantify a decision maker's influence on outcomes in a given environment. The multiplicity of indices attests to the fact that 'power' or 'influence' – often used as synonyms in the literature – have many facets and several meanings. We will mostly let the formal definitions of prominent indices, their probabilistic interpretations and some of their properties speak for themselves here. To readers who are curious about more philosophical discussions of power, the non-trivial distinction between 'power over' and 'power to', or links between influence and causation we recommend to look at Riker (1964) and, most comprehensively, Morriss (2002). Riker (1986) and Felsenthal and Machover (2005) provide illuminating accounts of the history of voting power analysis, and how essentially the same mathematical indicators have been re-invented several times from different backgrounds. Comprehensive introductions to power indices are given by Felsenthal and Machover (1998) and Laruelle and Valenciano (2008).

2 Basic Framework for Binary Voting

Many decisions that are taken by councils, committees, parliaments, etc. involve more than two alternatives, often very many. Most of the analysis of voting power has nonetheless focused on the *binary* case, and so will we. We assume that two actions are available to each voter; the voting rule links them to two distinct collective outcomes – for instance, a "yes" or "no" decision on some proposal, selection of candidate A or candidate B, etc. We will review possibilities for formalizing different voting rules in such a binary context before discussing indices of voting power in Section 3. An elaborate account of the binary voting framework is given by Taylor and Zwicker (1999). We point to analysis of richer settings in Section 5.

2.1 Simple voting games

The first ingredient to formalizing any given institutional arrangement is a comprehensive list of the relevant decision makers, interchangeably called voters, agents or players. This is typically done by collecting them in a finite set N. The set might contain the names of individuals, acronyms of parties, etc. but it is often convenient to simply use letters $A, B, C, \ldots$ or to work with numbers and the set $N = \{1, 2, \ldots, n\}$.

A binary voting rule can then be described by indicating for every subset S of N – reflecting different divisions of the voters between "yes" and "no", A and B, etc. – whether the members of S can jointly bring about their shared goal or not. This can be done by specifying a mapping $v\colon S \mapsto \{0, 1\}$, also called *characteristic function*. The statement $v(S) = 1$ indicates that the members of S can jointly succeed to impose their will. Analogously $v(S) = 0$ means that they cannot. Collections $S \subseteq N$ of voters are also called *coalitions*. Those with $v(S) = 1$ are referred to as *winning coalitions*, the others as *losing coalitions*.

An alternative to defining the function v is to simply list all coalitions of decision makers who are sufficient to pass the bill, elect the preferred candidate, etc. This amounts to specifying a set $\mathcal{W}$ of subsets of N which comprises all winning coalitions. For instance, the example in Section 1 could be formalized by letting $N = \{A, B, C\}$ denote the set of players and $\mathcal{W} = \{AB, BC, AC, ABC\}$ the corresponding set of winning coalitions (where AB is short-hand notation for N's subset $\{A, B\}$).

It is generally assumed that the empty set $\varnothing$ is a losing coalition ($v(\varnothing) = 0$), while the 'grand coalition' of all players is winning ($v(N) = 1$). Moreover, if a coalition S is already winning, it is usually required that any bigger coalition T which contains it is winning, too. Formally, this requires v to be *monotonic*: $S \subseteq T \Rightarrow v(S) \leq v(T)$. A combination (N, v) or $(N, \mathcal{W})$ which satisfies these assumptions is also called a *simple (voting) game*.

Monotonicity allows us to describe a simple voting game more efficiently by listing only the so-called *minimal winning coalitions*, i.e., by indicating the set $\mathcal{W}^m \subseteq \mathcal{W}$ of those winning coalitions S such that exit of any member $i \in S$ turns the remainder $S \setminus \{i\}$ into a losing coalition. In the example above, $\mathcal{W}^m = \{AB, BC, AC\}$. Of course, it is equivalent to define a simple game by providing a complete list $\mathcal{L}$ of its losing coalitions or, more concisely, the set $\mathcal{L}^M$ of *maximal losing coalitions*, i.e., those losing coalitions that would become winning if anyone joined. Here $\mathcal{L}^M = \{A, B, C\}$.

Real voting systems frequently have the property that if the members of a coalition S are winning, e.g., have the required majority for passing a bill, then the non-members are losing. The corresponding simple game is then called *superadditive* or *proper*: $S \in \mathcal{W} \Rightarrow N \setminus S \in \mathcal{L}$. However, some types of proposals – to establish a parliamentary inquiry commission, for instance – require less than a majority of votes. Then both S and its complement $N \setminus S$ can be winning, and the corresponding game is not proper. The opposite case in which both S and its complement $N \setminus S$ are losing often arises because of supermajority requirements (e.g., a two-thirds majority held neither by the government nor the opposition). When, in contrast, the complement of every losing coalition is winning, the game is called *strong*: $S \in \mathcal{L} \Rightarrow N \setminus S \in \mathcal{W}$. Simple games that are proper and strong are known as *constant sum* or *decisive*: $S \in \mathcal{W} \Leftrightarrow N \setminus S \in \mathcal{L}$.

2.2 Weighted voting games

Many binary voting rules have a convenient representation by means of *voting weights*. Specifically, a simple game is called a *weighted (voting) game* if one can find a non-negative number w_i for each voter $i \in N$ together with a positive *quota* q such that coalition S is winning if and only if the total weight $w(S) = \sum_{j \in S} w_j$ of its members meets or exceeds the quota. In this case, we write $[q; w_1, \ldots, w_n]$ interchangeably with (N, v) or $(N, \mathcal{W})$.

A given voting rule may not involve weights explicitly but can still correspond to a weighted voting game. For instance, decisions by the 15 members of the UN Security Council require nine "yes" votes in total and, moreover, none of the five permanent members must have cast a *veto* by voting "no". If we disregard – despite its importance in practice – the possibility of abstentions, this rule may be represented using the quota-and-weights combination $[39; 7, 7, 7, 7, 7, 1, \ldots, 1]$.

But not every simple voting game is weighted. To see this, consider the 6-player game defined by $N = \{A, B, C, D, E, F\}$ and the set of maximal losing coalitions $\mathcal{L}^M =$

$\{ACE, ACF, ADE, ADF, BCE, BCF, BDE, BDF\}$. Note first that coalitions $T_1 = ACE$ and $T_2 = BDF$ – being elements of $\mathcal{L}^M$ – are both losing. By contrast, $S_1 = AB$ and $S_2 = CDEF$ are both winning: neither is a subset of any element of $\mathcal{L}^M$. The former two coalitions involve the same players equally often as the latter, just in a different arrangement. Hence, $w(S_1) + w(S_2) = w(T_1) + w(T_2)$ would have to hold *if* a weighted representation existed. Such representation would need to involve a quota q so that $w(S_1) \geq q$ and $w(S_2) \geq q$, and hence $w(S_1) + w(S_2) \geq 2q$. Moreover, $w(T_1) < q$ and $w(T_2) < q$. But this contradicts $w(S_1) + w(S_2) = w(T_1) + w(T_2)$.[1] So this game is *not* weighted. Prominent practical examples of decision rules that do not correspond to a weighted voting game include the qualified majority rule of the Council of the EU, the amendment of the Canadian constitution, and also the US federal legislative system involving House, Senate, Vice-President, and President. There is no way to recurse to weights in order to assess voting power for them.

We remark that when one weighted representation of a simple game exists, many others automatically exist too. For instance, the winning and losing possibilities in our introductory example are represented equally well, e.g., by $[50.1\%; 48\%, 37\%, 15\%]$, $[50; 49, 48, 2]$ or $[2; 1, 1, 1]$. The latter is the *minimum sum integer representation* of the corresponding simple game $(N, \mathcal{W})$: among all weighted representations of the game which involve only integer numbers, its weight sum is the smallest possible. This arguably provides a very transparent way to describe voters' possibilities to form winning coalitions. It seems tempting to employ rescalings of the respective minimum integer weights as indicators of power (see Freixas and Kaniovski 2014). But, as mentioned, many important institutional rules do not allow for a weighted representation. Moreover, the minimum sum integer representation can fail to be unique when more than seven voters are involved (e.g., Kurz 2012). It is hence useful to know the tools discussed in the following section.

[1] Formally, the combination $\langle S_1, S_2; T_1, T_2 \rangle$ is known as a *2-trade*: we "trade" members between two winning coalitions so that both become losing. Existence of such trades implies non-weightedness. Conversely, a game must be weighted if no k-trade exists for large enough k (Taylor and Zwicker 1999, Thm. 2.4.2).

3 Power Indices for Binary Voting

As pointed out in Section 1, power and influence are multi-faceted. This makes them hard to quantify in an uncontroversial way, even for the sort of well-defined, quite restricted decision environments reflected by simple voting games. We refer to Bertini et al. (2013) for a long but still incomplete list of mappings from the space of n-player simple voting games to n-dimensional vectors of real numbers which have all been suggested to capture some aspect of voting power.

The plethora of available indices and mostly futile debates on whether this or that index is "better" can be intimidating. Some scholars have used it as an excuse to dismiss the whole approach – especially if they find the binary voting framework of Section 2 too restrictive anyway. We prefer to regard power indices as useful cousins to solution concepts in game theory. (Simple voting games may, in fact, be treated as a special class of cooperative games – those where the image of characteristic function v is restricted to $\{0, 1\}$.) So the following analogy, articulated by Aumann (1987), applies to them in equal measure:

> "Different solution concepts are like different indicators of an economy; different methods for calculating a price index; different maps (road, topo, political, geologic etc., not to speak of scale, projection, etc.); ... They depict or illuminate the situation from different angles; each one stresses certain aspects at the expense of others."

We will first discuss the two best-known indices of voting power, the *Penrose-Banzhaf index* (PBI) and the *Shapley-Shubik index* (SSI). They are particular weighted averages of the effects or *marginal contributions* of voters to all conceivable coalitions $S \subseteq N$. One alternative, which leads to the *Deegan-Packel index* (DPI) and *Holler-Packel index* (HPI), is to take only a particular class of coalitions into consideration. Another, pursued by the *nucleolus*, is based on a real or virtual surplus which the institution in question is to divide according to the given majority rule. Computing the expected surplus shares that are implied by the rule together with a plausible bargaining protocol constitutes a power evaluation quite distinct from assessments based on marginal contributions.

3.1 Penrose-Banzhaf index

The *Penrose-Banzhaf index* (PBI) of voter $i \in N$ in simple game (N, v) is formally defined by

$$PBI_i(N, v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} \Big[v(S \cup \{i\}) - v(S)\Big], \tag{1}$$

where n denotes the total number of voters. The summation considers each coalition S that does not yet include voter i, and checks if the addition of i to S would make a difference to outcomes.[2] The latter is the case if, under the institutional rules described by (N, v), proposals can be passed with the joint support of the members of S and of i, but not the support of S's members alone.

The bracketed term $[v(S \cup \{i\}) - v(S)]$ in equation (1) is known as the *marginal contribution* of voter i to coalition S. If it equals 1 in simple game (N, v), we say i's decision is *pivotal* or *crucial* or *critical* or *decisive* for the collective outcome (given the support of S); we also say that the pair $(S, S \cup \{i\})$ is a *swing* for i, or that i has a swing. The summation in equation (1) is thus a simple count of swing positions that may arise for voter i in the decision making body, and each of them is weighted by the factor $1/2^{n-1}$. This factor is the inverse of the total number 2^{n-1} of coalitions S which do not include voter i and so could potentially be swung by i. (It also equals the number of coalitions S which include i – see fn. 2.) So the PBI of voter i represents the ratio of i's actual to i's potential swing positions. One can also interpret this ratio as the probability of i being decisive *if all coalitions are assumed to be equally likely*. This approach to measuring voting power was first suggested by Lionel S. Penrose (1946, 1952), but forgotten for a while. The index became popular after its independent reinvention by John F. Banzhaf (1965).[3]

The PBI has several attractive features. For one, voter i's PBI is closely related to the expected *success* of i under the given voting rule. More specifically, there are two outcomes of a "yes"-or-"no" vote in which a given player i can be said to be

[2] Equivalently, one could consider all coalitions S which include i and check if the exit of i would make a difference, i.e., evaluate $\sum_{S \subseteq N, i \in S}[v(S) - v(S \setminus \{i\})]$. One may also sum simply over *all* subsets $S \subseteq N$ in either case, since $S \cup \{i\} = S$ if $i \in S$ and $S = S \setminus \{i\}$ if $i \notin S$. The respective totals always equal the number of winning coalitions in (N, v) that include i minus the number that do not include i.

[3] See Felsenthal and Machover (2005). The PBI was also independently reinvented by Rae (1969) and Coleman (1971). Rae ruled out the empty coalition $S = \varnothing$, while Coleman conditioned either on passage or rejection of a proposal. So both considered inessential re-scalings of equation (1).

successful: either i voted "yes" and the collective decision is "yes" (so some coalition S with $i \in S \in \mathcal{W}$ formed), or i voted "no" and the collective decision is "no" (S with $i \notin S \in \mathcal{L}$ formed). Under the mentioned probability interpretation of the PBI, i.e., considering all 2^n different coalitions $S \subseteq N$ to have equal probabilities $\Pr(S) = 1/2^n$ of being formed, the likelihoods of these two events combine to

$$\Pr\Big(\text{voter } i \text{ is successful in } (N, v)\Big) = \Pr(i \in S \in \mathcal{W} \text{ or } i \notin S \in \mathcal{L}) = \frac{1}{2} + \frac{1}{2} PBI_i(N, v). \quad (2)$$

(See, e.g., Felsenthal and Machover 1998, Thm. 3.2.16, for the calculation.) So under the probabilistic assumptions of the PBI, a voter's power and his expected success are almost the same thing – the corresponding quantitative measures are merely an affine transformation of each other. But this is true only when indeed all coalitions have equal probability. The latter is equivalent to assuming, first, that every voter is (in expectation) for or against a proposal *equally likely* and, second, the realization of each voter i's preference is *statistically independent* of that of any other voter j. This excludes common biases or partial dependencies among the voters.

Statistical independence and equiprobability define the *binomial voting model* from which Lionel S. Penrose concluded a counterintuitive but mathematically sound benchmark for the fair design of two-tier voting systems. Specifically, suppose that we are not taking a simple voting game as given but instead trying to find one for the following situation: m disjoint voter groups $N_1, \ldots, N_m$ with $n_1, \ldots, n_m$ individuals each decide on the same proposal within their groups by standard majority voting. Then, a delegate from each group $j \in \{1, \ldots, m\}$ represents the majority decision within N_j in an assembly of delegates, i.e., a game with player set $M = \{1, \ldots, m\}$. One may think of N_1, N_2, etc. as corresponding to citizens of a US federal state or EU member country, and the assembly M as the US Electoral College or Council of the EU. From a democratic fairness perspective, it is then desirable to conduct voting in the assembly in such a way that the *indirect influence* of any voter $i \in N = N_1 \cup \ldots \cup N_m$ on collective decisions is the same, no matter to which constituency j voter i belongs. So we search for a voting rule v such that (M, v) implements the influence aspect of the *one-person-one-vote principle* of democracy.

It is intuitive that the probability of voter i being decisive within his constituency j falls with the constituency's size n_j. Hence larger constituencies need bigger voting power in (M, v) in order to equalize individual voters' chances to be decisive at large. But, under the binomial model, it turns out that decisiveness within the constituency is *inversely proportional to the square root of j's population size* n_j, not inversely proportional to n_j.[4] It follows that the voting rule for the delegates should be selected such that their implied PBI values are proportional to the square root of the represented population sizes, not population sizes directly. This is known as *Penrose's square root rule*. Having it as an objective benchmark for the fair design of two-tier voting systems adds to the attractiveness of the PBI as a power measure. The rule's normative force for practical institutional design depends, of course, on the extent to which one believes (or, perhaps appealing to more basic normative principles, chooses to believe) in the underlying independence and equiprobability assumptions for voters' preferences.[5]

The PBI also satisfies a number of desirable mathematical properties, which the literature often refers to as *axioms*. For one, the index is *anonymous*: it treats players the same no matter what name or label is assigned to them – only their roles under the given collective decision rule matter. Namely, if we choose to describe a rule not by (N, v), but some (N, v') where the player identifiers are describing the same roles in a different order (so, formally, v' results from v by applying a permutation ρ to the player set N), then the same PBI values as before will be assigned to each role. The only difference is that a given role will be associated with player $i' = \rho(i)$ if it was previously held by i.

Another desirable feature of the PBI, and satisfied also by most other power indices, is the *null player property*. A voter $i \in N$ is called a *null player* (also a *dummy player*) in (N, v) if i's presence or absence never has an effect on the outcome, i.e., $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N$. The corresponding axiom is: an index shall assign zero power to every null player.

A third, somewhat less straightforward requirement is the so-called *transfer*

[4] This can be seen relatively easily as follows. Suppose the population size $n_j = 2k + 1$ is odd. Then decisiveness of voter i in constituency j requires that the $2k$ other voters in the constituency are evenly split and so i holds the balance of power. The probability of exactly k "yes"-votes, when "yes" and "no"-decisions are equally likely and independent, is $(2k)!/(k! \cdot k!) \cdot 1/2^k \cdot 1/2^k$. Applying Stirling's formula to the factorials, this approximately equals $\sqrt{2/(\pi \cdot n_j)}$ when n_j is big, i.e., it is proportional to $1/\sqrt{n_j}$.

[5] The dependence structure of voter preferences is the key issue in deriving design benchmarks for two-tier voting systems under a wide range of objectives. See, e.g., Felsenthal and Machover (1999) and Kirsch (2013) on avoidance of majoritarian paradoxes, Barberà and Jackson (2006) and Koriyama et al. (2013) on utilitarian welfare maximization, and Kurz et al. (2017) on equalization of influence.

property. It relates the power of any player i under four different voting rules for a fixed set of players: the first two, say, v and $\tilde{v}$, are arbitrary. The last two are constructed from the former as follows: a coalition S is winning in $v \wedge \tilde{v}$ – also called the *meet* of v and $\tilde{v}$ – if and only if it is winning *in v and in $\tilde{v}$*. So $v \wedge \tilde{v}$'s set of winning coalitions is the intersection $\mathcal{W} \cap \tilde{\mathcal{W}}$ of v's and $\tilde{v}$'s. Similarly, S is winning in $v \vee \tilde{v}$ – also called the *join* of v and $\tilde{v}$ – if and only if it is winning *in v or in $\tilde{v}$*. So $v \vee \tilde{v}$'s set of winning coalitions is the union $\mathcal{W} \cup \tilde{\mathcal{W}}$. The transfer property requires that the sum of each player i's power in v and $\tilde{v}$ be equal to the sum of i's power in $v \wedge \tilde{v}$ and $v \vee \tilde{v}$. The PBI indeed satisfies $PBI(v) + PBI(\tilde{v}) = PBI(v \wedge \tilde{v}) + PBI(v \vee \tilde{v})$.

In fact, the PBI can be shown to be the *only* power index that satisfies anonymity, null player property, transfer property and the 'scaling' condition that the index values of all players always add to the total number of swings divided by 2^{n-1}. This was shown by Dubey and Shapley (1979).[6] One can, therefore, read that the PBI is *axiomatically characterized* by these properties. Other sets of properties which are satisfied only by the PBI, and in this sense characterize it *uniquely*, exist too. See Laruelle and Valenciano (2001).

The particular scaling of the PBI implies that players' power values usually do not add up to 1. For instance, we obtain a PBI vector of $\left(\frac{11}{16}, \frac{5}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}\right)$ with sum $\frac{25}{16}$ for the weighted voting game $[51; 35, 20, 15, 15, 15]$. Sometimes it is convenient for comparisons of *relative power* to normalize the PBI such that it always sums to 1. This gives the *normalized Penrose-Banzhaf index*

$$nPBI_i(N, v) = \frac{PBI_i(N, v)}{\sum_{i \in N} PBI_i(N, v)}. \tag{3}$$

To make the distinction clear, the PBI in equation (1) is then also referred to as the *absolute PBI*. The normalization in equation (3) not only voids the (admittedly artificial) original scaling condition and the PBI's transfer property, but also destroys

[6] The key idea – also in analogous axiomatizations of other indices – is to initially consider so-called *unanimity games* v_T in which a fixed subset T of players from N constitutes the only minimal winning coalition; so $v_T(S) = 1 \Leftrightarrow S \supseteq T$. The null player, anonymity and scaling conditions directly fix the index for such games: it is 0 for the null players $i \notin T$. The t others must have the same index value by anonymity; scaling implies this to be 2^{1-t} and hence to coincide with the PBI of (N, v_T). The values for all unanimity games (N, v_T) are then extended to arbitrary simple games (N, v) by an induction argument which uses the transfer property, noting that $v = v_{T_1} \vee \ldots \vee v_{T_m}$ if (N, v)'s minimal winning coalitions are $\mathcal{W}^m = \{T_1, \ldots, T_m\}$.

any meaningful probabilistic interpretation of a voter's PBI. It should hence be applied with care.

3.2 Shapley-Shubik index

If the normalization to having a sum of 1 is imposed as a property – known as *efficiency* of an index – right from the start together with anonymity, the null player property and the transfer property (see Dubey 1975), then we arrive at the *Shapley-Shubik index* (SSI):

$$SSI_i(N,v) = \sum_{S \subseteq N} \frac{s! \cdot (n-s-1)!}{n!} \cdot \left[v(S \cup \{i\}) - v(S)\right] \tag{4}$$

$$= \sum_{S \subseteq N} \frac{(s-1)! \cdot (n-s)!}{n!} \cdot \left[v(S) - v(S \setminus \{i\})\right] \tag{5}$$

where s denotes the cardinality of coalition S and we define 0! to equal 1. Like the PBI, the SSI averages voters' marginal contributions. This can equivalently be stated in terms of joining or leaving any conceivable coalition, i.e., as (4) or (5) (see fn. 2).

In contrast to the PBI, the SSI does not give equal weight to all marginal contributions of voter i: the factor in front of the brackets is larger if i swings large or small coalitions than medium-sized ones. For instance, the factor in equation (4) is $1/5$ for $S = \emptyset$ and $S = N \setminus \{i\}$, but only $1/30$ if S includes $s = 2$ out of $n = 5$ voters. Such non-uniform weighting may seem curious at first. It arises in a relatively natural fashion, however, if we change the perspective from sets S of voters to orderings of the voters, i.e., distinguish between ABC and ACB, for instance.

An *ordering* corresponds to a permutation or mapping $\rho\colon N \to N$ where $\rho(i)$ indicates voter i's position between 1 and n when $N = \{1, \ldots, n\}$. The SSI assigns an equal weight to all $n! = n \cdot (n-1) \cdot \ldots \cdot 1$ orderings of the players, i.e., we can also write

$$SSI_i(N,v) = \frac{1}{n!} \sum_{\rho \in P} \left[v(S_{\rho(i)} \cup \{i\}) - v(S_{\rho(i)})\right] \tag{6}$$

where $S_{\rho(i)}$ denotes the subset of voters who have smaller positions than voter i according to a given ordering ρ, and P is the set of all conceivable orderings of the voters.

Motivations of why orderings of players rather than just their division into a "yes"-camp S and a "no"-camp $N \setminus S$ are of interest tend to leave the tight binary framework described in Section 2. The first proposals of the SSI, by Shapley (1953) for more general cooperative games and specifically for voting situations by Shapley and Shubik (1954), pointed out that coalition formation often takes place in a sequential fashion. Players reveal support for a new motion one by one, with those who are most enthusiastic about it presumably going first. The decision maker whose support first brings about the required majority, settling the issue, is the pivotal voter. His contribution comes with special influence: he is the least enthusiastic of the members required for passing, and thus the one who most needs to be won over by concessions on open details of the proposal. The SSI can thus be associated with an endogenous perspective on collective decisions, rather than imagining votes on exogenous take-it-or-leave-it proposals (made by some non-strategic outsider) to which voters would seem to respond like independent fair coins if we took the PBI's binomial model literally. With an endogenous take on decisions, it is plausible to assume that all orderings – reflecting sequential entry into the coalition by degree of enthusiasm or for other reasons – are equally likely a priori.

A similar motivation of the SSI involves an explicit account of policies and preferences over a real interval of alternatives. Suppose all voters have single-peaked preferences over these alternatives. So each voter i can be characterized by a particular ideal policy location λ_i inside the policy interval for whatever issue randomly turns up for decision. A simple majority of weight requirement, as in Section 1, then makes the most preferred policy of the (weighted) *median voter* the only one that cannot be beaten by any other policy proposal: for any proposal to the left of the median location, the median voter and the voters to his right command more than 50% of weight and will rather move right; for any proposal to the right, the median and voters to his left can block and move left.

The weighted median corresponds exactly to the decisive or pivotal player if we order voters according to their policy locations. The probability of voter i having a marginal contribution of 1, which is given by equation (6) for the case that each ordering is equally likely, is hence the same as that of ending up with the collective decision x^* which i likes best: $x^* = \lambda_i$. We can therefore view the SSI's assumption of each ordering having probability $1/n!$ as short-hand for the assumption that all voters pursue an ideal policy that is drawn independently from the same continuous distribution over an interval of alternatives and voter i's SSI as capturing his chances to be the institution's weighted median or, more generally, pivotal player.[7]

With this interpretation of the SSI as describing a continuous median voter environment in reduced form, it is possible to apply it to problems of fair design of voting systems, similar to the original purpose of the PBI. Namely, consider the two-tier voting setup introduced in Section 3.1 but let voters have single-peaked preferences over an interval of alternatives, not just over "yes" and "no". Let each delegate adopt the position of the respective constituency's median voter as in the binary model. Add the assumption that preferences are positively correlated within each constituency N_j while they remain independent across constituencies. Having some preference similarity within the constituencies is a plausible reason for why boundaries in a two-tier system cannot simply be redrawn such that one obtains $n_1 = \ldots = n_m$. The SSI of delegate j in voting game (M, v) then approximates the probability that the most-preferred policy of the median voter in constituency j becomes the collective decision. Since each voter in a given constituency N_j has a probability of $1/n_j$ to be N_j's median, *equal influence* for all individuals $i \in N_1 \cup \ldots \cup N_m$ requires SSI values for the delegates $1, \ldots, m$ which are *proportional to their represented population sizes* n_j, not their square roots. See Kurz et al. (2017) for details.

The different focuses of PBI and SSI – on equiprobable coalitions of supporters of an exogenous proposal vs. on equiprobable player orderings related to endogenous specifics of a proposal that is passed – generally result in different power indications for a given game, even if one invokes the normalized version of the PBI. For instance, the SSI vector of the weighted voting game $[51; 35, 20, 15, 15, 15]$ is $SSI(N, v) = \left(\frac{9}{20}, \frac{1}{5}, \frac{7}{60}, \frac{7}{60}, \frac{7}{60}\right)$, compared to $nPBI(N, v) = \left(\frac{11}{25}, \frac{1}{5}, \frac{3}{25}, \frac{3}{25}, \frac{3}{25}\right)$. The former

[7] See Black (1948) on the case of supermajority requirements. We note that other probabilistic ways to arriving at the coefficients in (4) or (5) exist. One is to directly assume that coalition sizes follow a particular probability distribution and that all coalitions of a given size have equal conditional probabilities. See Laruelle and Valenciano (2005). Alternatively, voters $1, \ldots, n$ can be assumed to support a random proposal in a conditionally independent fashion with respective probabilities $p_1, \ldots, p_n$. If all probabilities p_i are equal to the same number p and this number is drawn uniformly from the unit interval $[0, 1]$, then one arrives at the SSI. This *homogeneity assumption* for individual acceptance probabilities is contrasted by Straffin (1988) with the *independence assumption* of choosing each p_i independently from the uniform distribution on $[0, 1]$, which yields the PBI. Also see Owen (1995, ch. 12) on this issue and his concept of the *multilinear extension* of a simple game (N, v).

ascribes more power to the largest voter and less to the three small voters, compared to the latter.

It is worthwhile to mention, however, that SSI and PBI usually rank voters in the same way. More precisely, a player i in a simple game (N, v) can be called (weakly) *more desirable* than player j if the replacement of j by i in a coalition S which contains j but not i never lowers the winning status of S. See Isbell (1956). Formally, we write $i \succsim j$ if $v(\{i\} \cup S \setminus \{j\}) \geq v(S)$ for each $S \subseteq N \setminus \{i\}$ with $j \in S$. (Note that $i \succsim j$ and $j \succsim i$ might hold simultaneously. For instance, this is true for all three voters in the introductory example, or for voters 3, 4 and 5 in the weighted game above.) Both PBI and SSI respect this desirability relation among players. Namely, $i \succsim j$ in a given simple game (N, v) implies both $SSI_i(N, v) \geq SSI_j(N, v)$ and $PBI_i(N, v) \geq PBI_j(N, v)$. The only situations where two voters i and j may be ranked differently by PBI and SSI arise when i and j are not comparable, that is, neither $i \succsim j$ nor $j \succsim i$ holds. This is not possible for any weighted voting game.[8] In particular, both PBI and SSI are *locally monotonic* indices: if voter i's weight and voter j's weight satisfy $w_i \geq w_j$ in any arbitrary representation of a given weighted voting game, then the indicated power of i is at least as big as that of j.

3.3 Deegan-Packel index and Holler-Packel index

This monotonicity property is not necessarily satisfied when a power index evaluates only a subset of coalitions in order to ascribe power to players. Deegan and Packel (1978) and Holler and Packel (1983), for instance, have proposed to include only the set $\mathcal{W}^m$ of *minimal winning coalitions* (MWC) when assessing power in simple games. The *Deegan-Packel index* (DPI) is formally defined as

$$DPI_i(N, v) = \frac{1}{|\mathcal{W}^m|} \sum_{S \in \mathcal{W}^m,\, i \in S} \frac{1}{s} \tag{7}$$

[8] The class of simple games in which $i \succsim j$ or $j \succsim i$ holds for every two players $i, j \in N$ are known as *complete simple games*. PBI and SSI are *ordinally equivalent* for complete simple games (cf. Freixas 2010), and since $w_i \geq w_j$ implies $i \succsim j$, so are they for all weighted voting games. One of the rare examples of small incomplete games in which PBI and SSI ordinally differ is defined by the meet $v = v_1 \wedge v_2$ of $v_1 = [2; 1, 1, 1, 1, 0, 0, 0, 0]$ and $v_2 = [3; 0, 0, 0, 0, 2, 1, 1, 1]$. The PBI ranks players 1–4 higher than 6–8 while the SSI does the opposite. See Straffin (1988).

where s again denotes the number of elements of the coalition S in question. It can be viewed as assuming that each MWC $S \in \mathcal{W}^m$ arises with an equal probability and that members of S divide spoils related to winning equally among them (or, alternatively, that each member of S decides for the group with equal probability).

The *Holler-Packel index* (HPI) of voter i is the normalization

$$HPI_i(N,v) = \frac{h_i(N,v)}{\sum_{i \in N} h_i(N,v)} \quad \text{of} \quad h_i(N,v) = \sum_{S \in \mathcal{W}^m,\, i \in S} 1. \tag{8}$$

It equals i's share of swings in the MWC of (N,v). In contrast to the DPI, the spoils or power that go with being part of a coalition $S \in \mathcal{W}^m$ are not divided but treated as *non-rival* among its members, i.e., as having features of a public good or a club good of S. It is therefore also referred to as the *public good index*.

Membership to MWC need *not* relate monotonically to voting weight. For instance, the weighted voting game $[51; 35, 20, 15, 15, 15]$ comes with the set $\mathcal{W}^m = \{AB, ACD, ACE, ADE, BCDE\}$. The small voters C, D and E are members to 3 MWC, which is less than the 4 MWC that include A but more than the 2 MWC of mid-sized voter B. The corresponding HPI vector is $\left(\frac{4}{15}, \frac{2}{15}, \frac{3}{15}, \frac{3}{15}, \frac{3}{15}\right)$. As $w_B > w_C$ but $HPI_B(N,v) < HPI_C(N,v)$, this index – like the DPI – fails to be locally monotonic. A fortiori, it also fails to respect Isbell's desirability ranking $\gtrsim$ of voters.

We omit more detailed discussion here of whether such non-monotonicities and other so-called *paradoxes* of voting power should be regarded as "a serious pathology" (Felsenthal and Machover 1998, p. 246), which disqualifies DPI and HPI as plausible indicators of voting power or not. Some decision environments or measurement objectives may call for a focus on MWC. See, e.g., the discussion by Brams and Fishburn (1995) of Riker's "size principle" or Braham and van Hees (2009) on links between voting power and causation. Moreover, even if some scholars have serious qualms about the DPI and HPI, the observation that they yield non-monotonic power ascriptions only in a few games – such as the one above – may tell us something interesting about these games, something that would go unnoticed without considering the indices.

3.4 Nucleolus

Minimal winning coalitions also turn out to play a special role in non-cooperative game-theoretic models of bargaining, when committees decide and vote on the division of a fixed surplus. Following Baron and Ferejohn (1989), Snyder et al. (2005) and other authors have considered negotiations in which some agent $i \in N$ is randomly 'recognized' to propose a surplus division. This proposal is implemented if a winning coalition $S \in \mathcal{W}$ defined by the underlying decision rule (N, v) votes for it; otherwise a new proposer is drawn. The corresponding (stationary subgame-perfect) equilibrium strategies involve proposals such that the recognized player suggests a positive surplus share only for members of a MWC and himself. This is the most economical way to get approval for one's proposal. A player's expected equilibrium payoff generally depends on the assumed probability of being recognized as next proposer. Montero (2006) has discovered that there are recognition probabilities with the property that they equal the expected equilibrium surplus shares which arise for the given voting rule (N, v). They essentially coincide with the *nucleolus* of (N, v).

The nucleolus is a game-theoretic solution concept that was introduced by Schmeidler (1969) and first applied to weighted voting games by Peleg (1968). As Montero (2005) and Le Breton et al. (2012) explain in more detail, the bargaining foundations of the nucleolus make it a compelling indicator of power when voting pertains to endogenous surplus divisions.[9] García-Valiñas et al. (2016) have confirmed this empirically for the EU.

The nucleolus satisfies anonymity, the null player property and is efficient. (It follows that it must violate the transfer property; cf. Section 3.2.) In contrast to the power indices introduced in Sections 3.1–3.3, there is no closed formula by which the nucleolus can be calculated for arbitrary simple games. The nucleolus instead is defined as the solution to a minimization problem. Specifically, taking a simple game (N, v) as given and writing $x(S) = \sum_{i \in S} x_i$, we call a real vector $x = (x_1, \ldots, x_n)$ with $x_i \geq 0$ and $x(N) = 1$ an *imputation*. For any coalition $S \subseteq N$ and imputation x, the difference

[9] We note that there also exist links between equilibrium payoffs in non-cooperative models of bargaining and the SSI. The corresponding bargaining protocols have a less natural feel, however, than those for the nucleolus. For instance, it is assumed that only the random recognition as next proposer is linked to voting rule v, while acceptance of a proposal requires *unanimity* independently of v. See Laruelle and Valenciano (2008, ch. 4).

$e(S, x) = v(S) - x(S)$ is known as the *excess* of S at x. It can be interpreted as quantifying the coalition's dissatisfaction and potential opposition to an agreement on allocation x: the members of S can create a total surplus of $v(S)$ if they agree on collaborating (1 if S is winning), but imputation x only gives them $x(S)$. Now, for any fixed x, let $S_1, \ldots, S_{2^n}$ be an ordering of all coalitions such that the excesses at x are weakly decreasing, and denote these ordered excesses by $E(x) = \big(e(S_k, x)\big)_{k=1,\ldots,2^n}$. Intuitively speaking, $E(x)$ lists the potential dissatisfaction with a proposed surplus allocation x for all conceivable coalitions from largest dissatisfaction to smallest. We say that an imputation x is *lexicographically less* than an imputation y if $E_k(x) < E_k(y)$ for the smallest component k with $E_k(x) \neq E_k(y)$. The *nucleolus* of (N, v) is then defined as the lexicographically minimal imputation: among all conceivable divisions of the surplus of 1 that the grand coalition can generate, it selects the one where the maximal dissatisfaction across different subsets of N is as small as possible. If several imputations satisfy this requirement, it selects one of them such that also the second-highest dissatisfaction is minimal, and so forth. The required minimizations can be achieved by *linear programming*. See, e.g., Maschler et al. (2013, ch. 20).[10]

4 Further Aspects of Voting Power

4.1 Computation of power indices

The most straightforward way of computing pivotality-based indices like the PBI or SSI is *direct enumeration*. This means that one goes through one coalition or player ordering after another, checking each time if any player is decisive and keeping track of these incidences. Several tools that do this for weighted games with a moderate number of voters can be accessed online for free (type, e.g., "compute voting power index online" into a search engine). The tools require the simple voting game (N, v) of interest to have a representation as the meet of a small number of weighted games.[11] Such representation need not be known or may not even exist (cf. Kurz and Napel 2016). In this case, one can easily implement the enumeration method by oneself if one

[10] A free Mathematica package "TuGames" by Holger Meinhardt allows to calculate the nucleolus – and much more. See http://library.wolfram.com/infocenter/MathSource/5709.

[11] The smallest number d such that $v_1 \wedge \ldots \wedge v_d = v$ and $v_1, \ldots, v_d$ are all weighted games with the same player set N is known as the *dimension* of simple game (N, v). Online tools usually expect $d \leq 3$.

remembers that each integer number has a unique binary representation, involving only zeros and ones. It can be obtained by a single conversion command in many programming languages. Membership of players in a coalition can be expressed with zeros and ones, too. For instance, the subset $S = \{1,3,5\}$ of $N = \{1,2,3,4,5\}$ has the *indicator* or *characteristic vector* $x = (1,0,1,0,1)$ in which $x_i = 1$ indicates that $i \in S$ and $x_i = 0$ that $i \notin S$. This vector – 10101, for short – is the binary equivalent of 21 ($= 1+4+16$). Enumerating all coalitions $\emptyset, \{1\}, \ldots, N$ thus amounts to looping through the binary versions of numbers $0, 1, \ldots, 2^n - 1$.

Sometimes weighted games involve many symmetric players whose voting weights give rise to relatively few distinct weight sums. Whenever $w(S)$ assumes considerably fewer different values than there are coalitions S, it pays to work with the *generating function method* instead of enumerating coalitions. The key idea is to cleverly count the number of all coalitions that come with a particular aggregate weight $\hat{w}$ and to evaluate pivotality in all of them in one go. Here it helps greatly that one can determine the cardinality $b_{\hat{w}} = |\{S\colon w(S) = \hat{w}\}|$ of the set of all coalitions S with $\sum_{i\in S} w_i = \hat{w}$ without constructing this set explicitly. Namely, the numbers $b_{\hat{w}}$ can be obtained by expanding, i.e., multiplying out, the factors in the *generating function*

$$f(x) = \prod_{j=1}^{n}(1 + x^{w_j}) = (1 + x^{w_1}) \cdot (1 + x^{w_2}) \cdot \ldots \cdot (1 + x^{w_n}) \tag{9}$$

$$= (1 + x^{w_1} + x^{w_2} + x^{w_1+w_2}) \cdot \ldots \cdot (1 + x^{w_n}) = \ldots = \sum_{\hat{w}=0}^{w(N)} b_{\hat{w}} \cdot x^{\hat{w}}. \tag{10}$$

The coefficients $b_{\hat{w}}$ result from collecting all terms which involve the same exponent of x, which can be done with a single command in many mathematical toolboxes. Various extensions and algorithmic refinements of the method have been developed since Mann and Shapley (1962) first applied it to the US Electoral College. See, e.g., Lindner (2004, ch. 11) and Alonso-Meijide et al. (2012).

Mann and Shapley had earlier resorted to the *Monte Carlo approximation method*. This considers only a randomly drawn sample of coalitions or player orderings, rather than all. One evaluates players' pivot frequencies in the sample, and uses these frequencies as unbiased estimates of the corresponding pivot probabilities of the players. We refer to Leech (2003) for further approximation methods.

4.2 Limit results for weighted voting rules

A key raison d'être for power indices is the generally non-linear relationship between a voter's weight and power. This non-linearity applies independently of whether one associates voting power with decisiveness on exogenous take-it-or-leave-it proposals, with influence on proposals selected from a continuous scale, or with rent shares linked to budget decisions. The mathematical reason is the discrete and combinatorial nature of forming coalitions: a voter's weight is either added to the tally in full, or not at all.

Intuitively, the lumpiness of voting weights should matter less, the more voters are involved. Possibly, the stylized examples considered in Sections 1 and 3 just involve too few players for voting power to be proportional to voting weight? This suspicion is partially correct. Namely, the ratio of the voting powers of two voters i and j converges to the ratio w_i/w_j of their voting weights if one considers ever larger decision making bodies and certain 'regularity conditions' are met. One might interpret this as saying that there is ever smaller loss in failing to distinguish between voting weight and power when larger decision making bodies are concerned. This is broadly confirmed by calculations for the Council of the EU, the US Electoral College, or the IMF's Board of Governors.

However, the mentioned regularity requirements are important. For instance, if we divide 49% of total weight among $n-1$ small voters while a single big voter keeps a weight of 51% regardless of n, then, assuming a simple majority quota, the latter agent has 100% of voting power. No matter how big an n we consider, there are $n-1$ null players and one *dictator player*, who is characterized by the fact that a coalition $S \subseteq N$ wins if and only if this player is a member. No convergence of power ratios to weight ratios takes place.

So a first requirement for expecting proportionality of weight and power in the limit, i.e., for ever larger voting bodies, is that every individual voter's weight should become negligible in relation to the total weight sum.[12] Secondly, one needs to rule

[12] This is not to say that large games are non-interesting if a few major players, say, $1, \ldots, k$ maintain a non-negligible voting weight, while those of voters $k+1, k+2, \ldots$ vanish in an 'ocean' of small players who share the remaining weight. The situation at stockholders' meetings of publicly traded companies often corresponds to such an *oceanic (voting) game*. Shapiro and Shapley (1978) and Dubey and Shapley (1979) have shown how the voting power of the k major players can be determined by analyzing voting only among them. This is much easier than doing calculations that involve all players. For instance,

out rare cases where no limit of the ratio of voting powers exists. The latter is, for instance, the case for the (admittedly contrived) sequence of n-player weighted voting games where player 1 has weight $w_1 = 1$ while $w_i = 2$ for all $i = 2, \ldots, n$, with a simple majority quota of $q = (2n-1)/2$. Whenever n is even – for instance, with $n = 4$ and hence a total weight of seven – $n/2$ big players are necessary and sufficient to achieve a majority. Then 1 is a null player, with zero power for all mentioned indices. However, when n is odd – moving, say, to $n = 5$ and a total weight of nine – player 1 has exactly the same possibilities to bring about a majority as the other players: any coalition of $(n+1)/2$ voters wins. It follows that 1's relative voting power is $1/n$ just like that of voters 2, 3, etc. So the ratio of voter 1's and 2's voting power alternates between 0 and 1 forever. In particular, it does not converge to the weight ratio of 1/2.

At a technical level, such problems can be avoided by focusing on voting games that have a replicative structure as we increase n. This means that all voters are supposed to come from a finite set of 'prototypes'. Each of the voter prototypes is characterized by a particular absolute voting weight. The total relative weight of each prototype is assumed to be non-vanishing as $n \to \infty$. Under these conditions, *limit theorems* establish that voting weight and a particular voting power index indeed become proportional as $n \to \infty$. Neyman (1982) and Lindner and Machover (2004) have established this for the SSI and PBI; Kurz et al. (2014) for the nucleolus. Findings for SSI and nucleolus apply to all decision quotas short of unanimity, while the corresponding *Penrose limit theorem* assumes a simple majority threshold. We are unaware if analogous results exist for the DPI or HPI.

4.3 The inverse problem

The a priori distribution of power in a voting body can, of course, be evaluated not only for a single rule but many. One may then want to select the rule – say, voting weights and a quota – which gives rise to an especially desirable power distribution. For example, PBI values proportional to the square roots of population sizes or SSI values proportional to population sizes have particular equitability properties in the context of two-tier voting systems (see Sections 3.1 and 3.2). The challenge is to find a voting

the PBI's binomial model implies that the 'ocean' will essentially be split half-half between "yes" and "no". So the PBI for the big players can be computed from a game that only involves players $1, \ldots, k$, their weights, and the original quota q reduced by half of the aggregate weight of the oceanic players.

game (N, v) which induces a power vector, such as normalized PBI values $nPBI_i(N, v)$ for $i \in \{1, \ldots, n\}$, as close as possible to a particular target vector $p^* = (p_1^*, \ldots, p_n^*)$. This is known as the *inverse (power) problem* for the considered index.

Suppose, in accordance with Penrose's square root rule (Section 3.1), that we choose the normalized square roots of population sizes in n constituencies as our target p^*. We would ideally implement a voting rule v such that $nPBI_i(N, v) = p_i^*$ holds for all constituencies $i \in N$. Provided that we stick to weighted voting rules, or simple voting games more generally, this is too much to ask for, however.[13] The reason is the discreteness of voting which we highlighted in Section 4.2. The number of different simple games, related to the *Dedekind numbers* in mathematics, grows so fast in a voting body's size that it is currently still unknown for $n \geq 9$ members. But it is finite, while there are infinitely many conceivable choices for p^*. It would be pure coincidence if a game existed that comes with exactly the desired power distribution p^*.

One therefore strives to obtain a solution to the minimization problem

$$\min \ d(nPBI(N, v); p^*) \ \text{ such that } v \in G \tag{11}$$

where G describes the space of permissible games – all simple games, only complete simple games (see fn. 8), or only weighted games – and $d(\cdot)$ formalizes a particular notion of *distance*, i.e., what it means to be 'close to p^*'. We could, for instance, use the Euclidean metric $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ in order to obtain minimal quadratic deviations from our target.

Solving problem (11) is not easy. Exact solutions can be obtained for up to 9 voters by enumerating the elements of G if the latter includes only weighted or complete simple games. For larger player numbers, *integer linear programming* techniques can be applied. See Kurz (2012). However, the corresponding algorithms need not terminate in reasonable time, so that heuristic approaches may actually be preferable for $n \geq 12$ (e.g., Kurz and Napel 2014). These can produce an exact hit by chance. And, generally speaking, the limit theorems discussed in Section 4.2 imply that errors from, first,

[13] A larger class of voting rules would, for instance, allow the decision quota q to be drawn randomly before each decision. Then one could implement any normalized target distribution p^* for the normalized PBI *exactly* by choosing $w_i = p_i^*$ for all $i \in N$ and drawing the quota q from a uniform distribution over the interval $(\frac{1}{2}, 1]$. See Dubey and Shapley (1979, Cor. 3). Such randomized decision rules, however, do not seem to play any role in practice.

guessing that weights proportional to the target p^* are optimal and, second, using hill-climbing techniques to find local improvements tend to become very small for 'regular' problems when n is sufficiently big.

5 Concluding Remarks

It should be clear from the above that assessments of a priori voting power are a non-trivial exercise already when binary decisions are concerned or represent bargaining and median voter environments in reduced form. Things get both more realistic and complicated when richer sets of feasible inputs and outputs of a voting process are considered explicitly, when additional procedural structure is imposed, or when one wants to incorporate institutional features which imply non-symmetric relations between the players.

For instance, several variations of power indices have been proposed for situations where the decision makers are grouped into disjoint *a priori unions*, i.e., subsets of voters who always join or leave a coalition S as a bloc after internal deliberation, or when their interaction involves *restricted communication*, so that a coalition S can only include A and C if also B is present as perhaps their unique connection under a given *communication structure*. The best-known indicators of power in voting games with a priori unions are the *Owen index* and the *Banzhaf-Owen index* (Owen 1977, 1982). The *Myerson value*, the *restricted Banzhaf value* and the *position value* have, among others, been proposed for voting games with restricted communication. Respectively see Myerson (1977), Owen (1986) and Borm et al. (1992). Power indices for the case when coalition formation depends on gradual communication intensities between voters, which can reflect preferences for coalescing derived from data in *a posteriori analysis of voting power*, have been proposed by Aleskerov (2006).

Various authors have also dealt with situations in which voters can choose between more than two options. For instance, Felsenthal and Machover (1997) and Laruelle and Valenciano (2012) have investigated *ternary* and *quaternary dichotomous voting rules*. There, collective decisions are still binary but voters have the option to "abstain" in addition to voting "yes" or "no", or even to "abstain" and to "not participate". The standard notions of decisiveness or pivotality can readily be extended to such rules. But the particular a priori probability distribution which is underlying an index becomes harder to justify. For instance, is it compelling to assume that the marginal probability for "abstain" is the same as that for, respectively, a "yes" and a "no" vote?

Freixas and Zwicker (2003) have considered (j, k) *simple games*. In these, each voter can choose between j ordered levels of approval of a motion or candidate – e.g., different grades on a scale from "A" to "F". The voting rule then maps each profile of approval levels which is submitted by the decision makers to one of k ordered outcomes such as final grades from "pass with distinction" to "fail"; or "make an attractive offer", "make an offer", "don't make an offer". *Multicandidate voting games*, in which each player votes for exactly one of k unordered candidates and the given rule selects a single winner, have been analyzed for example by Bolger (1986).

The above-mentioned extensions of the basic binary framework of voting do, however, not account for well-known problems and the strategic component of voting whenever there are *three alternatives or more*. See, e.g., Nurmi (1987, 1999).

Strategic aspects to voting power become particularly important when the decision making process is structured by a given step-by-step *voting procedure*. For instance, the *ordinary legislative procedure* of the EU involves agenda setting, several amendment stages, and a bargaining or conciliation stage with well-defined roles for the European Parliament, the Commission and the Council. It is also called the *codecision procedure* of the EU because of the supposedly coequal roles assigned to Parliament and Council. Do the rules really make them equally influential a priori? Standard indices of voting power fail to reflect the sequential and strategic nature of legislative processes into which votes on endogenous proposals are usually embedded. This means the PBI's or SSI's power indications may be quite misleading if procedures and strategic reasoning matter not only ex post but also from an a priori perspective, i.e., considering institutional rules while ignoring elusive historical preference data. As the saying goes, if all you have is a hammer, everything looks like a nail. So it is no surprise that simple binary power indices have been applied to environments in which they do not make sense. This has prompted some scholars to call for "a moratorium on the proliferation of index-based studies" (Garrett and Tsebelis 2001, p. 100).[14]

Voting power indices like PBI, SSI, DPI, etc. that are based on marginal contributions can – with appropriate scaling – be regarded as measuring the expected

[14]For a staunch defense of index-based studies see Felsenthal et al. (2003).

sensitivity of the collective decision to preferences and actions of a given voter. Adopting this perspective of *power measurement as sensitivity analysis*, it is possible to extend analysis of voting power from binary, non-procedural and non-strategic environments to voting processes with richer inputs and outputs, procedural aspects and strategic interaction. A more detailed case for this is made by Napel and Widgrén (2004). The applicable techniques and modeling options are illustrated, for instance, by Maaser and Mayer (2016) for the EU's ordinary legislative procedure.

Napel and Widgrén (2011) compare the results from a strategic voting power assessment of decision making by the European Commission, Parliament and members of the Council with those from standard index-based studies. They find that the latter can serve as rather good first approximations even if some error cannot be avoided.

Suppose that a sound first approximation is sufficient. The reader may then still be left with the question: which of the many available indices should I use? All caveats which are implied by the discussion above notwithstanding, a rough-and-ready recommendation could be as follows. First, it is both adequate and computationally most practical to use the PBI for collective decisions on proposals that are binary in nature and made by an external agent who cannot or will not react to preferences of the considered voters (i.e., proposals are not selected strategically from a bigger, non-binary set of options). When votes are instead cast on endogenous proposals, determined directly by the voting body in question or indirectly by an agenda setter who reacts to preferences of the decision makers, then the SSI suggests itself for preferences or enthusiasms for change which vary on some left–right, high–low, etc. spectrum. For committees that primarily vote on endogenous divisions of rents or surpluses, the appealing bargaining foundations of the nucleolus make this the most pertinent tool. If none of these cases applies or one wants to directly obtain more than a first approximation of how power is distributed in a given decision making environment, more detailed game-theoretic analysis is recommended. This can, of course, still be a priori in the sense of extending the uniform probability assumptions regarding preferences for "yes" or "no", or preference-induced player orderings, to the applicable payoff structure on richer domains.

References

Aleskerov, F. (2006). Power indices taking into account agents' preferences. In B. Simeone and F. Pukelsheim (Eds.), *Mathematics and Democracy*, pp. 1–18. Berlin: Springer.

Alonso-Meijide, J. M., J. Freixas, and X. Molinero (2012). Computation of several power indices by generating functions. *Applied Mathematics and Computation 219*(8), 3395–3402.

Aumann, R. (1987). Game theory. In J. Eatwell, M. Milgate and P. Newman (Eds.), *The New Palgrave: A Dictionary of Economics*, pp. 460–482. London: MacMillan.

Banzhaf, J. F. (1965). Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review 19*(2), 317–343.

Barberà, S. and M. O. Jackson (2006). On the weights of nations: Assigning voting weights in a heterogeneous union. *Journal of Political Economy 114*(2), 317–339.

Baron, D. P. and J. A. Ferejohn (1989). Bargaining in legislatures. *American Political Science Review 83*(4), 1181–1206.

Bertini, C., J. Freixas, G. Gambarelli, and I. Stach (2013). Comparing power indices. *International Game Theory Review 15*(2), 1340004-1–19.

Black, D. (1948). The decisions of a committee using a special majority. *Econometrica 16*(3), 245–261.

Bolger, E. M. (1986). Power indices for multicandidate voting games. *International Journal of Game Theory 15*(3), 175–186.

Borm, P., G. Owen, and S. H. Tijs (1992). On the position value for communication situations. *SIAM Journal on Discrete Mathematics 5*(3), 305–320.

Braham, M. and M. van Hees (2009). Degrees of causation. *Erkenntnis 71*(3), 323–344.

Brams, S. J. and P. C. Fishburn (1995). When is size a liability? Bargaining power in minimal winning coalitions. *Journal of Theoretical Politics 7*(3), 301–316.

Coleman, J. S. (1971). Control of collectivities and the power of a collectivity to act. In B. Lieberman (Ed.), *Social Choice*, pp. 269–300. New York, NY: Gordon and Breach.

Deegan, J. and E. W. Packel (1978). A new index of power for simple n-person games. *International Journal of Game Theory 7*(2), 113–123.

Dubey, P. (1975). On the uniqueness of the Shapley value. *International Journal of Game Theory 4*(3), 131–139.

Dubey, P. and L. Shapley (1979). Mathematical properties of the Banzhaf power index. *Mathematics of Operations Research 4*(2), 99–131.

Felsenthal, D. and M. Machover (1999). Minimizing the mean majority deficit: The second square-root rule. *Mathematical Social Sciences 37*(1), 25–37.

Felsenthal, D. S., D. Leech, C. List, and M. Machover (2003). In defence of voting power analysis. *European Union Politics 4*(4), 473–497.

Felsenthal, D. S. and M. Machover (1997). Ternary voting games. *International Journal of Game Theory 26*(3), 335–351.

Felsenthal, D. S. and M. Machover (1998). *The Measurement of Voting Power – Theory and Practice, Problems and Paradoxes*. Cheltenham: Edward Elgar.

Felsenthal, D. S. and M. Machover (2005). Voting power measurement: A story of misreinvention. *Social Choice and Welfare 25*(2), 485–506.

Freixas, J. (2010). On ordinal equivalence of the Shapley and the Banzhaf values for cooperative games. *International Journal of Game Theory 39*(4), 513–527.

Freixas, J. and S. Kaniovski (2014). The minimum sum representation as an index of voting power. *European Journal of Operational Research 233*(3), 739–748.

Freixas, J. and W. S. Zwicker (2003). Weighted voting, abstention, and multiple levels of approval. *Social Choice and Welfare 21*(3), 399–431.

García-Valiñas, M., S. Kurz, and V. Zaporozhets (2016). Key drivers of EU budget allocation: Does power matter? *European Journal of Political Economy 43*, 57–70.

Garrett, G. and G. Tsebelis (2001). Even more reasons to resist the temptation of power indices in the EU. *Journal of Theoretical Politics 13*(1), 99–105.

Holler, M. J. and E. W. Packel (1983). Power, luck and the right index. *Zeitschrift für Nationalökonomie (Journal of Economics) 43*(1), 21–29.

Isbell, J. R. (1956). A class of majority games. *Quarterly Journal of Mathematics 7*(1), 183–187.

Kirsch, W. (2013). On Penrose's square-root law and beyond. In M. J. Holler and H. Nurmi (Eds.), *Power, Voting, and Voting Power: 30 Years After*, pp. 365–387. Berlin: Springer.

Koriyama, Y., J.-F. Laslier, A. Macé, and R. Treibich (2013). Optimal apportionment. *Journal of Political Economy 121*(3), 584–608.

Kurz, S. (2012). On the inverse power index problem. *Optimization 61*(8), 989–1011.

Kurz, S., N. Maaser, and S. Napel (2017). On the democratic weights of nations. *Journal of Political Economy* (forthcoming).

Kurz, S. and S. Napel (2014). Heuristic and exact solutions to the inverse power index problem for small voting bodies. *Annals of Operations Research 215*(1), 137–163.

Kurz, S. and S. Napel (2016). Dimension of the Lisbon voting rules in the EU Council: A challenge and new world record. *Optimization Letters 10*(6), 1245–1256.

Kurz, S., S. Napel, and A. Nohn (2014). The nucleolus of large majority games. *Economics Letters 123*(2), 139–143.

Laruelle, A. and F. Valenciano (2001). Shapley-Shubik and Banzhaf indices revisited. *Mathematics of Operations Research 26*(1), 89–104.

Laruelle, A. and F. Valenciano (2005). Assessing success and decisiveness in voting situations. *Social Choice and Welfare 24*(1), 171–197.

Laruelle, A. and F. Valenciano (2008). *Voting and Collective Decision-Making*. Cambridge: Cambridge University Press.

Laruelle, A. and F. Valenciano (2012). Quaternary dichotomous voting rules. *Social Choice and Welfare 38*(3), 431–454.

Le Breton, M., M. Montero, and V. Zaporozhets (2012). Voting power in the EU Council of Ministers and fair decision making in distributive politics. *Mathematical Social Sciences 63*(2), 159–173.

Leech, D. (2003). Computing power indices for large voting games. *Management Science 49*(6), 831–838.

Lindner, I. (2004). *Power Measures in Large Weighted Voting Games: Asymptotic Properties and Numerical Methods*. Ph. D. thesis, University of Hamburg. http://ediss.sub.uni-hamburg.de/volltexte/2004/2222/pdf/Dissertation.pdf.

Lindner, I. and M. Machover (2004). L. S. Penrose's limit theorem: Proof of some special cases. *Mathematical Social Sciences 47*(1), 37–49.

Maaser, N. and A. Mayer (2016). Codecision in context: Implications for the balance of power in the EU. *Social Choice and Welfare 46*(1), 213–237.

Mann, I. and L. S. Shapley (1962). Values of large games, VI: Evaluating the Electoral College exactly. Memorandum RM-3158-PR, The Rand Corporation.

Maschler, A., E. Solan, and S. Zamir (2013). *Game Theory*. Cambridge: Cambridge University Press.

Montero, M. (2005). On the nucleolus as a power index. *Homo Oeconomicus 22*(4), 551–567.

Montero, M. (2006). Noncooperative foundations of the nucleolus in majority games. *Games and Economic Behavior 54*(2), 380–397.

Morriss, P. (2002). *Power: A Philosophical Analysis* (2nd ed.). Manchester: Manchester University Press.

Myerson, R. (1977). Graphs and cooperation in games. *Mathematics of Operations Research 2*(3), 225–229.

Napel, S. and M. Widgrén (2004). Power measurement as sensitivity analysis: A unified approach. *Journal of Theoretical Politics 16*(4), 517–538.

Napel, S. and M. Widgrén (2011). Strategic vs. non-strategic voting power in the EU Council of Ministers: The consultation procedure. *Social Choice and Welfare 37*(3), 511 – 541.

Neyman, A. (1982). Renewal theory for sampling without replacement. *Annals of Probability 10*(2), 464–481.

Nurmi, H. (1987). *Comparing Voting Systems*. Dordrecht: D. Reidel Publishing (Kluwer).

Nurmi, H. (1999). *Voting Paradoxes and How to Deal with Them*. Berlin: Springer.

Owen, G. (1977). Values of games with a priori unions. In R. Henn and O. Moeschlin (Eds.), *Mathematical Economics and Game Theory*, pp. 76–88. Berlin: Springer.

Owen, G. (1982). Modification of the Banzhaf-Coleman index for games with a priori unions. In M. J. Holler (Ed.), *Power, Voting, and Voting Power*, pp. 232–238. Würzburg: Physica-Verlag.

Owen, G. (1986). Values of graph-restricted games. *SIAM Journal on Algebraic and Discrete Methods 7*(2), 210–220.

Owen, G. (1995). *Game Theory* (3rd ed.). San Diego, CA: Academic Press.

Peleg, B. (1968). On weights of constant-sum majority games. *SIAM Journal on Applied Mathematics 16*(3), 527–532.

Penrose, L. S. (1946). The elementary statistics of majority voting. *Journal of the Royal Statistical Society 109*(1), 53–57.

Penrose, L. S. (1952). *On the Objective Study of Crowd Behaviour*. London: H. K. Lewis & Co.

Rae, D. W. (1969). Decision rules and individual values in constitutional choice. *American Political Science Review 63*(1), 40–56.

Riker, W. H. (1964). Some ambiguities in the notion of power. *American Political Science Review 58*(2), 341–349.

Riker, W. H. (1986). The first power index. *Social Choice and Welfare 3*(4), 293–295.

Schmeidler, D. (1969). The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics 17*(6), 1163–1170.

Shapiro, N. Z. and L. S. Shapley (1978). Values of large games, I: A limit theorem. *Mathematics of Operations Research 3*(1), 1–9.

Shapley, L. S. (1953). A value for *n*-person games. In H. W. Kuhn and A. W. Tucker (Eds.), *Contributions to the Theory of Games*, Volume II, pp. 307–317. Princeton, NJ: Princeton University Press.

Shapley, L. S. and M. Shubik (1954). A method for evaluating the distribution of power in a committee system. *American Political Science Review 48*(3), 787–792.

Snyder, Jr., J. M., M. M. Ting, and S. Ansolabehere (2005). Legislative bargaining under weighted voting. *American Economic Review 95*(4), 981–1004.

Straffin, Jr., P. D. (1988). The Shapley-Shubik and Banzhaf power indices as probabilities. In A. E. Roth (Ed.), *The Shapley Value – Essays in Honor of Lloyd S. Shapley*, pp. 71–81. Cambridge: Cambridge University Press.

Taylor, A. D. and W. S. Zwicker (1999). *Simple Games*. Princeton, NJ: Princeton University Press.