THE PREDICTION VALUE*

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Abstract: We introduce the prediction value (PV) of player i as the difference between the conditional expectations of $v(S)$ when i cooperates or not in a probabilistic TU game. The latter combines a standard TU game and a probability distribution over the set of coalitions. The PV reflects the importance of information about a given player's behavior for predicting, e.g., committee decisions that are subject to opinion interdependencies. The PV is characterized by anonymity, linearity, a consistency requirement and two normalization conditions. Every multinomial probabilistic value, hence every binomial semivalue, coincides with the PV for a particular family of probability distributions. So the PV can be regarded as a power index in specific cases. Conversely, some semivalues – including the Banzhaf but not the Shapley value – can be interpreted in terms of informational importance.

Keywords: influence, voting games, cooperative games, Banzhaf value, Shapley value.

JEL-classification: C71, D72, D89

1. Introduction

Concepts of power and importance in models of cooperation are central to numerous studies in sociology, political science, mathematics, and economics. Much of the literature applies values or power indices which attribute fixed roles – often perfectly symmetric – to all players in the underlying coalition formation process and then focus on their *marginal contributions*. Most prominent examples are the *Shapley value* and *Banzhaf value* (Shapley 1953; Banzhaf 1965); others can be found in Roth (1988), Owen (1995), Felsenthal and Machover (1998) or Laruelle and Valenciano (2008b). The values differ in how marginal contributions to distinct coalitions are weighted.

With an appropriate rescaling, weights on specific marginal contributions can always be interpreted as a probability distribution. So Shapley value, Banzhaf value, and more generally *probabilistic values* (Weber 1988) correspond to the *expectation of a difference*. This difference

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Date: December 5, 2016.

*The authors thank an associate editor and an anonymous referee for helpful comments. They also acknowledge helpful feedback from seminar audiences in Bayreuth, Bielefeld, Bilbao, Jena and Turku as well as from participants and three anonymous referees of the Sixth Workshop on Cooperative Games in Multiagent Systems 2015, Istanbul.

is between the worth of a random coalition S which is drawn from $2^{N\setminus i}$ according to a value-specific probability distribution P_i and the worth of the same coalition when i joins. That is, a probabilistic value equals $\mathbb{E}_{P_i}[v(S \cup i) - v(S)]$ for a fixed family of distributions $\{P_i\}_{i \in N}$.[1]

Unless the probabilistic value in question is also a (multibinary or) *multinomial probabilistic value* (cf. Puente 2000; Freixas and Puente 2002; Carreras and Puente 2015a), the presence of player i in the realized coalition S can statistically depend on whether player $j \neq i$ belongs to it or not. This may plausibly be the case, for example, when voting is preceded by a process of information transmission or opinion formation.[2] Unfortunately the expectation $\mathbb{E}_{P_i}[v(S \cup i) - v(S)]$, interpreted as power or importance of player i, can behave in strange ways when the family of distributions $\{P_i\}_{i \in N}$ implicates correlated behavior. The following example illustrates the conceptual problem.[3]

Example 1. Consider the canonical simple majority decision rule with an assembly of 5 voters. Let P be the probability distribution that assigns probability 0 to the 20 coalitions containing exactly two or exactly three voters; and equal probability of $1/12$ to each of the remaining 12 coalitions. Irrespective of whether we derive P_i by projecting P to $2^{N\setminus i}$ or by conditioning P on $i \notin S$, the probabilistic value $\mathbb{E}_{P_i}[v(S \cup i) - v(S)]$ is zero for all players. However, that *no* member of this decision body should have any voting power or importance is somewhat counterintuitive.

Here, the expectation of a difference is uninformative since non-zero marginal contributions $v(S \cup i) - v(S) > 0$ do not count when the underlying probability distribution P treats $S \cup i$ and S as null events. More generally, the problem with the expected marginal contribution $\mathbb{E}_{P_i}[v(S \cup i) - v(S)]$ is that it treats i's decision, say, to change her *no* vote into a *yes* (or vice versa) as being fully detached from the respective probabilities of observing the considered two coalitions with and without i.

This paper proposes an alternative approach: namely, to consider the *difference of two expectations*. These expectations will be derived from a given probabilistic description P of coalition formation. The latter plays the same role as $\{P_i\}_{i \in N}$ does for probabilistic values or corresponding families $\{P_i^v\}_{i \in N}$ do for values that evaluate marginal contributions in game v-specific ways.[4] However, we take P as a primitive of the collective decision situation under investigation, rather than of the solution concept.

We thus depart from most of the previous literature in two respects: first, similar to Laruelle and Valenciano (2008a), we explicitly consider *probabilistic games* (N, v, P) where (N, v) is a standard TU game and P is a probability distribution on N's power set 2^N. Second, we introduce a new value that reflects the difference between two conditional expectations. Specifically, we

[1] It is equivalent to consider suitable probability distributions P_i on $\{S \in 2^N : i \in S\}$ and then to evaluate $\mathbb{E}_{P_i}[v(S) - v(S \setminus i)]$. We adopt the usual notational simplifications like writing $S \setminus i$ or $S \cup ij$ instead of $S \setminus \{i\}$ or $S \cup \{i, j\}$.

[2] See, for example, the seminal opinion formation model of DeGroot (1974): individuals start with initial opinions (beliefs) on a subject represented by an n-dimensional vector of probabilities, and repeatedly update their individual opinion based on the current opinions of their peers. Different structures of consensus formation can be captured by different network topologies.

[3] We owe this example to Moshé Machover.

[4] This is, for instance, the case when positive probability is attached only to *minimal winning coalitions* (see, e.g. Holler 1982 and Holler and Li 1995).

define the *prediction value (PV)* of any given player $i \in N$ as the difference in v's expected value when the distribution $P|i$ which conditions P on the event $\{i \in S\}$ and the distribution $P|\neg i$ which conditions on $\{i \notin S\}$ are applied. In other words, we suggest to evaluate $\mathbb{E}_{P|i}[v(S)] - \mathbb{E}_{P|\neg i}[v(S)]$ instead of $\mathbb{E}_{P_i}[v(S \cup i) - v(S)]$. The two coincide in interesting special cases, but not in general. In particular, our approach is similar in spirit but factually differs from the evaluation of conditional decisiveness as in Laruelle and Valenciano (2005, 2008a).

The margin between the respective conditional expectations can be interpreted as the importance of a player in the probabilistic game (N, v, P) in several ways. Most generally, it captures the informational or predictive value of knowing i's decision in advance of the process which divides N into some final coalition S and its complement. Moreover, in case i's membership of the coalition which supports a specific bill or cooperates in a joint venture is statistically independent of others, the PV provides a measure of i's influence on the outcome of collective decision making, or of i's power in (N, v, P).

The existing literature – the references above are a small selection – obviously does not suffer from a shortage of solution concepts in general, nor of ones targeted at quantifying a player's power in TU games. But as Aumann (1987) argued: "Different solution concepts are like different indicators of an economy; different methods for calculating a price index; different maps (road, topo, political, geologic etc., not to speak of scale, projection, etc.); ...They depict or illuminate the situation from different angles; each one stresses certain aspects at the expense of others." Taking up Aumann's metaphor, we do *not* suggest another scale or projection of a player's vector of marginal contributions here. We rather propose to look at a slightly different kind of map. For some situations – involving statistically independent cooperation decisions – the picture may look identical to that generated by, say, the Banzhaf value (giving the latter additional force/basis/clout); in others, such as Example 1, it will provide a new perspective.

Example 1 (continued). In the considered assembly, the probability of coalition S was $P(S) = 0$ for $|S| = 2$ or $|S| = 3$ and $P(S) = 1/12$ otherwise. The respective conditional probabilities follow from Bayes' rule. In particular, knowing that player i cooperates (i.e., is part of S) gives rise to

$$P|i(S) = \begin{cases} \frac{1}{6} & \text{if } S = \{i\} \text{ or } S = N \backslash j, j \neq i \text{ or } S = N, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Similarly, we obtain

$$P|\neg i(S) = \begin{cases} \frac{1}{6} & \text{if } S = \varnothing \text{ or } S = \{j\}, j \neq i \text{ or } S = N \backslash i, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

in case i is known *not* to cooperate. The expected value of $v(S)$, which corresponds to the probability that the assembly passes the proposal in question, is changed by the observation that, say, voter 1 supports rather than opposes it by

$$\begin{aligned} \mathbb{E}_{P|1}[v(S)] - \mathbb{E}_{P|\neg 1}[v(S)] &= \sum_{S \ni 1} v(S) \cdot P|1(S) - \sum_{T \not\ni 1} v(T) \cdot P|\neg 1(T) \\ &= \frac{5}{6} - \frac{1}{6} = \frac{2}{3}. \end{aligned} \tag{3}$$

The information that player 1 votes in a particular way is not enough to predict an assembly decision perfectly. (That would be the case if $\mathbb{E}_{P|1}[v(S)] - \mathbb{E}_{P|\neg 1}[v(S)] = 1$.) But it changes the odds significantly. Player 1 may not have voting power in the traditional sense, but his or her vote is important from an informational perspective. It may be quite valuable to investors or speculators, for instance. The same applies here to the other members of the assembly to identical degrees. Asymmetric voting weights or asymmetric roles in opinion dynamics and coalition formation would naturally give rise to different PV numbers for the respective assembly members.

A null player who has a voting weight that cannot matter for matching a required majority threshold *and* whose behavior is uncorrelated with the remaining players has a PV of zero. Endowing the same player with greater voting weight will at some point translate into a positive value – reflecting the difference that her vote can now make for the outcome, just like traditional indices. But leaving initial voting weights unchanged, the PV will also ascribe positive importance to the null player if interdependencies make its cooperation a predictor of whether a proposal is passed.

Plausible causes for dependencies abound and, for instance, include the possibility that the player in question is actually without vote but 'followed' by the official voters as, say, their paramount leader. The proposed change of perspective – from, traditionally, the expected difference that a player would make by an ad-hoc change of coalition membership towards the difference in expectations for the collective outcome which is associated with that player's cooperation – opens the route to studying voting and coalition formation as the result of social interaction. Final votes may be determined by whether i is initially a supporter or opponent even if i is a null player of (N, v), and this is arguably a source of power just like official voting weight. We believe that evaluating changes in conditional expectations can help to quantify this in future research.

Here, we introduce and investigate properties of the prediction value. We formally define it in Section 3, after collecting some preliminaries in Section 2. We describe a set of characteristic properties in Section 4 and relate the PV to traditional probabilistic values in Section 5. The considered distributions P could embody the *a prioristic* presumptions of traditional power measures, i.e., be the uniform distribution on 2^N or the space of permutations on N. (Interestingly, the latter does *not* make PV and Shapley value coincide.) But P could equally well be based on empirical data – say, observations of past voting behavior in a decision making body like the US Congress, EU Council, etc. We briefly conduct such *a posteriori analysis* with the PV in an application to the Dutch Parliament in Section 6, and we conclude in Section 7. All mathematical proofs are contained in the Appendix.

2. Preliminaries

A *TU game* is an ordered pair (N, v) where $N \subset \mathbb{N}$ represents a non-empty, finite set of players and $v \colon 2^N \to \mathbb{R}$ is the characteristic function which specifies the worth $v(S)$ of any subset or coalition $S \subseteq N$ and satisfies $v(\varnothing) = 0$. The set of all TU games is denoted by $\mathcal{G}$, and the set of all TU games with player set N by $\mathcal{G}^N$.

$(N, v) \in \mathcal{G}$ is a *simple game* if v is a monotone Boolean function, i.e., $v(S) \in \{0, 1\}$ and $v(S) \leqslant v(S')$ for all $S \subseteq S' \subseteq N$, such that $v(\varnothing) = 0$ and $v(N) = 1$. A coalition S with $v(S) = 1$ is then called *winning*. Given any non-empty coalition $S \subseteq N$, the so-called *unanimity game* u_S is defined by $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise. We will drop the player set N from our notation when it is clear from the context; so u_S is shorthand for (N, u_S). Moreover, we refer to $u_{\{i\}}$ simply as u_i.

Player i is called a *dummy player* in (N, v) if $v(S \cup i) - v(S) = v(i)$ for all $S \subseteq N \setminus i$. Every player i with $v(i) = 0$ is said to be *dependent*. A dummy player who is dependent is also known as a *null player*. If (N, v) is a simple game and $v(S) = 1$ if and only if $i \in S$, then player i is called a *dictator*.

TU games (N, v) have explicitly been combined with probability distributions P over coalitions $S \subseteq N$ since Owen (1972). P is typically mentioned in order to provide probabilistic motivation or foundations for a particular solution concept, not as a characteristic of a collective decision making situation. We want to emphasize its role as a primitive and define a *probabilistic game* as an ordered triple (N, v, P), where (N, v) is a TU game and P is a probability distribution on the power set of N, 2^N. The set of all probabilistic games is denoted by $\mathcal{PG}$; and $\mathcal{PG}^N$ is the restriction to the class of probabilistic games with player set N.

A *TU value* is a function which assigns a real number to all elements of N for any given TU game. An *extended value* is a mapping φ that assigns to each probabilistic game (N, v, P) a vector $\varphi(N, v, P) \in \mathbb{R}^{|N|}$. $\varphi_i(N, v, P)$ will be interpreted as a measure of the 'difference', in an abstract sense, that player i makes for the probabilistic game (N, v, P). It might, for instance, relate to the average of marginal contributions $v(S \cup i) - v(S)$ that are made by i to coalitions $S \in N \setminus i$, to the difference that i makes to a potential function (i.e., a mapping from $\mathcal{PG}$ to $\mathbb{R}$) when i is added to the player set N' such that $N' \cup i = N$, or to any other indicator of how important the behavior or presence of player i might be to the members of N or an outside observer.

TU values and extended values are defined on two distinct domains, $\mathcal{G}$ and $\mathcal{PG}$. Extended values can be regarded as technically the more general concept because any given TU value can be turned into an extended value simply by ignoring the distribution P that is specified as part of (N, v, P). For instance,

$$\phi_i(N, v, P) = \sum_{S \not\ni i} \frac{|S|!(|N| - |S| - 1)!}{|N|!}(v(S \cup i) - v(S)) \tag{4}$$

correspondingly 'extends' the Shapley value and the (extended) Banzhaf value can be defined by

$$\beta_i(N, v, P) = \frac{1}{2^{n-1}} \sum_{S \not\ni i}(v(S \cup i) - v(S)).^5 \tag{5}$$

Both the original Shapley TU value and the Banzhaf TU value (which was at first restricted to simple games, and later extended to general TU games by Owen 1975) are special instances of *probabilistic values*, as introduced by Weber (1988).[6] They are defined by

$$\Psi_i(N, v, Q) = \sum_{S \ni i} Q_i(S)(v(S) - v(S \setminus i)) = \mathbb{E}_{Q_i}[v(S) - v(S \setminus i)] \tag{6}$$

[5]When the considered set of players N is clear from the context, we simplify notation by writing $\sum_{S \not\ni i}$ instead of $\sum_{S \subseteq N: i \notin S}$, or $\sum_{S \ni i}$ instead of $\sum_{S \subseteq N: i \in S}$.

[6]Weber (1988) originally defined a "probabilistic value" for each individual player $i \in \{1, \ldots, n\}$ and referred to the corresponding n-vector as a *group value*. We follow the later terminology of Monderer and Samet (2002).

such that each element Q_i of the collection $Q = \{Q_i\}_{i \in N}$ denotes a probability distribution on $\{S \subseteq 2^N : i \in S\}$, or

$$\Psi_i(N, v, Q') = \sum_{S \not\ni i} Q'_i(S)(v(S \cup i) - v(S)) = \mathbb{E}_{Q'_i}[v(S \cup i) - v(S)] \tag{7}$$

such that Q'_i denotes a probability distribution on $2^{N \setminus i}$.[7]

Several subclasses of probabilistic values have received special attention. *Semivalues* satisfy (6) and (7) for weights $Q_i(S)$ and $Q'_i(S)$, respectively, which are identical for all $i \in N$ and depend on S only via its cardinality |S| (Dubey et al. 1981). Then for all $i \in N$

$$\Psi_i(N, v, Q') = f_i^q(N, v) = \sum_{S \subseteq N \setminus i} q_{|S|} \cdot \Big(v(S \cup i) - v(S)\Big) \tag{8}$$

for a vector of n non-negative numbers $q = (q_0, \ldots, q_{n-1}) \neq 0$ with

$$\sum_{k=0}^{n-1} \binom{n-1}{k} q_k = 1. \tag{9}$$

The Shapley value arises by setting $q_k = \frac{1}{n\binom{n-1}{k}}$; the Banzhaf index for $q_k = \frac{1}{2^{n-1}}$. The latter – but not the former – is also a *binomial semivalue*: there exists $0 < p < 1$ such that (8) holds for

$$q_k = p^k (1-p)^{n-k-1} \quad \text{for } k = 0, \ldots, n-1. \tag{10}$$

See Dubey et al. (1981), Carreras and Freixas (2008), and Carreras and Puente (2012).

Multinomial values have been introduced by Puente (2000) and are obtained from (6) or (7) by requiring that each player j is part of the formed coalition with probability $\tilde{p}_j$ independently of any other player.[8] Let $g_i^{\tilde{p}}(N, v)$ denote the multinomial value of player i for a fixed vector $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_n) \in [0, 1]^n$. Then for all $i \in N$

$$\Psi_i(N, v, Q') = g_i^{\tilde{p}}(N, v) = \sum_{S \subseteq N \setminus i} \prod_{j \in S} \tilde{p}_j \prod_{j \in N \setminus (S \cup i)} (1 - \tilde{p}_j) \cdot \Big(v(S \cup i) - v(S)\Big). \tag{11}$$

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For a given probabilistic game (N, v, P) define the conditional probability distributions $P|i$ and $P|\neg i$ as follows: for all $S \subseteq N$

$$P|i(S) = \begin{cases} \frac{P(S)}{\sum_{T \ni i} P(T)} & \text{if } i \in S \text{ and } \sum_{T \ni i} P(T) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

[7]The argument (N, v, Q) in (6) and (7) is more general than a probabilistic game. The latter follows as the special case in which each Q_i stems from the same probability distribution Q.

[8]Carreras and Puente (2015a) illustrate possibilities to connect probabilities $\tilde{p}_1, \ldots, \tilde{p}_n$ to political positions on a left-to-right axis. See Giménez et al. (2014) and Carreras and Puente (2015b) for applications of multinomial values to partnership formation and coalition structures in cooperative games. Properties of multinomial values, such as their monotonicity with respect to v, are studied by Domènech et al. (2016). Calvo and Santos (2000), among others, discuss further subclasses of probabilistic values such as *weighted Shapley values*, *weak semivalues* or *weighted weak semivalues*.

and, similarly,

$$P|\neg i(S) = \begin{cases} \frac{P(S)}{\sum_{T \not\ni i} P(T)} & \text{if } i \notin S \text{ and } \sum_{T \not\ni i} P(T) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Laruelle and Valenciano (2005) have suggested to consider the conditional decisiveness measures

$$\Phi_i^+(N, v, P) = \mathbb{E}_{P|i}[v(S) - v(S \setminus i)] \tag{14}$$

and

$$\Phi_i^-(N, v, P) = \mathbb{E}_{P|\neg i}[v(S \cup i) - v(S)], \tag{15}$$

as indicators of player i's importance in (N, v, P).[9] It is easy to see that $\Phi^+(N, v, P) = \Phi^-(N, v, P) = \beta(N, v, P)$ if and only if $P(S) \equiv 2^{-|N|}$ for all $S \subseteq N$, and one can similarly obtain identity with the (extended) Shapley value. Namely,

$$\Phi^+(N, v, P) = \phi(N, v, P) \iff P(\varnothing) = 0,\ P(S) = \frac{1}{s\binom{n}{s}\sum_{t=1}^{n}\frac{1}{t}} \text{ if } S \neq \varnothing, \tag{16}$$

$$\Phi^-(N, v, P) = \phi(N, v, P) \iff P(N) = 0,\ P(S) = \frac{1}{(n-s)\binom{n}{s}\sum_{t=1}^{n}\frac{1}{t}} \text{ if } S \neq N \tag{17}$$

with $n = |N|$ and $s = |S|$ (see Laruelle and Valenciano 2005, Prop. 3).[10]

We propose an altogether different approach to assessing the importance of N's members in a probabilistic game (N, v, P). It is *not* based on probabilistic values, nor marginal contributions in general. Weighted marginal contributions may misrepresent player i's importance in (N, v, P) in that they implicitly treat i's decision, say, to change her *no* vote into a *yes* (or vice versa) as being fully detached from the respective probabilities of observing the considered two coalitions with and without i. Example 1 already highlighted the effect that non-zero marginal contributions $v(S \cup i) - v(S) > 0$ do not count when the underlying probability distribution P treats both events $S \cup i$ and S as null events. Adding up marginal contributions also leads to strange conclusions if only one of the coalitions S and $S \cup i$ has positive probability.

Example 2. Consider an assembly of 3 voters in which coalitions $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$ are winning. Assume voters 2 and 3 are enemies and always vote contrary to each other. Here, coalition $S = \{1, 2\}$ might have positive probability under P and $P|\neg 3$, while $P(N) = 0$. A problem for measures like $\Phi^-(N, v, P)$ is then that they are strictly increased by a contribution which 3 makes in the null event of joining $S = \{1, 2\}$.

One thing that outside observers, members $j \neq i$ of N or i herself might still care about in the examples is the informational gain that comes with the knowledge: "i will (not) be part of the eventually formed coalition". Knowing this might imply that j cannot (or must) be amongst the members of the coalition. This may have ramifications for the expected surplus that is created or the passage probability of the bill being debated. In other words, it may be useful to base one's evaluation of collective decision making as described by (N, v, P) on $P|i$ rather than P when i is known to support the decision. This suggests looking at the difference $\mathbb{E}_{P|i}[v(S)] - \mathbb{E}_P[v(S)]$ as

[9] Laruelle and Valenciano (2008a) refer to $\Phi_i^+(N, v, P)$ as player i's "interim expected marginal contribution" and identify its relations with different classes of probabilistic values.

[10] Note that the respective distributions P which need to be assumed in order to obtain the Shapley value as a conditional expected marginal contribution in (16) and (17) differ.

a way of quantifying i's effect on the outcome. And it is arguably of similar interest – and may yield a rather different quantification of the difference that i's decision makes – not to look at how much i's support increases the expected worth $v(S)$ but at how much i's opposition lowers it, i.e., $\mathbb{E}_P[v(S)] - \mathbb{E}_{P|\neg i}[v(S)]$. Combining these two evaluations of how knowledge of i's decision changes the expectation of the game by summing them, we obtain:

Definition 1. The *prediction value (PV)* of player i in the probabilistic game (N, v, P) is defined as

$$\begin{aligned} \xi_i(N, v, P) &= \mathbb{E}_{P|i}[v(S)] - \mathbb{E}_{P|\neg i}[v(S)] \qquad (18) \\ &= \sum_{S \ni i} v(S) \cdot P|i(S) - \sum_{T \not\ni i} v(T) \cdot P|\neg i(T). \end{aligned}$$

Remark 1. In case that coalition membership is statistically independent for every $i \neq j$, i.e., if P is a product measure on 2^N, the equality $P|i(S) = P|\neg i(S \setminus i)$ holds whenever $i \in S$. Then equations (14), (15), and (18) all evaluate to the same number – for instance, to the Banzhaf value if $P(S) \equiv 2^{-|N|}$. That the "expectation of a difference" in (14) or (15) coincides with the "difference between two expectations" in (18), however, fails to hold in general. In particular, we will show in Corollary 2 that there is no probability distribution P which would allow the Shapley value to be interpreted as measuring informational importance.

Remark 2. The restriction of the prediction value to simple games has been identified by Häggström et al. (2006) as playing a key role in extending the *Condorcet jury theorem* (on asymptotically correct simple majority decisions by n statistically independent voters) to weighted majority decisions with arbitrary joint vote distributions.[11] Häggström et al. call the difference between expectations based on $P|i$ and $P|\neg i$ the "effect" of voter i. Their observation that player i's effect can be interpreted as a normalized form of the correlation between i's vote and the random jury outcome extends straightforwardly to probabilistic TU games. Namely, writing $\mathbf{1}_{\{i \in S\}}$ for the indicator function of event $\{i \in S\}$ and $p_i = P(\{S : i \in S\})$, the covariance of $v(S)$ and $\mathbf{1}_{\{i \in S\}}$ is

$$\begin{aligned} \mathrm{Cov}\big(v(S), \mathbf{1}_{\{i\in S\}}\big) &= \mathbb{E}_P[[v(S) \cdot \mathbf{1}_{\{i\in S\}}] - \mathbb{E}_P[v(S)] \cdot \mathbb{E}_P[\mathbf{1}_{\{i\in S\}}] \qquad (19) \\ &= \mathbb{E}_P\Big[v(S) \cdot \big(\mathbf{1}_{\{i\in S\}} - \mathbb{E}_P[\mathbf{1}_{\{i\in S\}}]\big)\Big] \\ &= p_i \mathbb{E}_P\Big[v(S) \cdot (1-p_i) \big| i \in S\Big] + (1-p_i)\mathbb{E}_P\Big[-p_i v(S) \big| i \notin S\Big] \\ &= p_i(1-p_i) \cdot \xi_i(N, v, P). \qquad (20) \end{aligned}$$

Dividing by standard deviations $\sigma_{v(S)}$ and $\sigma_{\mathbf{1}_{\{i\in S\}}} = \sqrt{p_i(1-p_i)}$, the PV and the correlation coefficient $\mathrm{Corr}\big(v(S), \mathbf{1}_{\{i\in S\}}\big)$ can be seen to satisfy

$$\xi_i(N, v, P) = \mathrm{Corr}\big(v(S), \mathbf{1}_{\{i\in S\}}\big) \cdot \frac{\sigma_{v(S)}}{\sqrt{p_i(1-p_i)}}. \qquad (21)$$

[11] Neeman (2014) further extended the analysis to weighted plurality decisions.

4. CHARACTERIZING THE PREDICTION VALUE

We will now provide an axiomatic characterization of the prediction value. We begin with two classical conditions that are part of many axiomatic systems in the literature on TU values. The first is *anonymity*, which requires that the indicated difference to the game that is ascribed to any player by an extended value does not depend on the labeling of the players. The second is *linearity*, which demands of an extended value that it is linear in the characteristic function component v of probabilistic games.

Definition 2. Consider two probabilistic games $G = (N, v, P)$ and $G' = (N', v', P')$ related through a bijection $\pi\colon N \to N'$ such that for all $S \subseteq N$, $v(S) = v'(\pi S)$ and $P(S) = P'(\pi S)$ where $\pi S := \{\pi(i)|i \in S\}$. An extended value φ is *anonymous* if for every such G and $G' \in \mathcal{PG}$

$$\varphi_i(N, v, P) = \varphi_{\pi(i)}(N', v', P') \text{ for all } i \in N. \tag{22}$$

Definition 3. An extended value φ is *linear* if for all $(N, v, P), (N, v', P) \in \mathcal{PG}$ and real constants α, β

$$\varphi\left(N, \alpha v + \beta v', P\right) = \alpha\varphi\left(N, v, P\right) + \beta\varphi\left(N, v', P\right). \tag{23}$$

Linearity combines two properties, *scale invariance* and *additivity*. Especially the latter is far from being innocuous.[12] But linearity is frequently imposed on solution concepts for TU games; and the PV, as the difference of two expectations, embraces it rather naturally.

The third characteristic property of the PV concerns the way how the respective extended values of two games G and G' compare when one can be viewed as a reduced form of the other.

Definition 4. Given $G = (N, v, P) \in \mathcal{PG}$ and a dependent player $i \in N$, the probabilistic game $G_{-i} = (N_{-i}, v_{-i}, P_{-i}) \in \mathcal{PG}$ is a *reduced game* derived from G by removal of i if $N_{-i} = N\backslash i$, and for all $S \subseteq N \setminus i$

$$P_{-i}(S) = P(S) + P(S \cup i) \tag{24}$$

$$v_{-i}(S) = \begin{cases} \frac{P(S)}{P(S)+P(S\cup i)} \cdot v(S) + \frac{P(S\cup i)}{P(S)+P(S\cup i)} v(S \cup i) & \text{if } P_{-i}(S) > 0,^{13} \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

So, when one moves from a given probabilistic game G to the reduced game G_{-i}, first, player i is removed from the set of players; second, the probabilities of all coalitions in G which only differ concerning i's presence are aggregated; and, third, the corresponding new worth $v_{-i}(S)$ of coalitions $S \subseteq N_{-i}$ is the convex combination of the associated old worths, $v(S)$ and $v(S \cup i)$, weighted according to the respective probabilities under P. We will require that the extended value of any player $j \in N_{-i}$ stays unaffected by the removal of i.[14]

Definition 5. An extended value φ is *consistent* if for all $G = (N, v, P) \in \mathcal{PG}$ and all dependent players $i \in N$ in v, we have $\varphi_j(G) = \varphi_j(G_{-i})$ for all $j \in N \setminus i$.

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[12] See, e.g., Felsenthal and Machover (1998, 6.2.26) and Luce and Raiffa (1957, p. 248).

[13] This definition of $v_{-i}(S)$ differentiates our reduction operation from the one in Laruelle and Valenciano (2008a, p. 76), which allows to interpret the expected values of v and v_{-i} in terms of a *potential function* in special cases (cf. Hart and Mas-Colell 1989). Note that if i were not a dependent player, i.e., $v(i) \neq 0$, then v_{-i} would not be a well-defined TU game because $v_{-i}(\varnothing) \neq 0$ in this case.

[14] The condition is vaguely reminiscent of the *amalgamation* properties considered by Lehrer (1988) or Casajus (2012).

One reason for why this consistency property could be desirable is the following. Suppose that the considered model is misspecified in the sense that a player of interest in the game is not taken into account by the rest of the players (or an outside observer). For instance, consider the situation of a voting game $G' = (N', v', P')$, where the presence of a lobbyist i has been neglected. The more accurate model would include the lobbyist and be $G = (N' \cup i, v, P)$. The effect of the lobbyist endorsing a proposal or opposing it would explicitly be captured by the probability distribution P: for example, voters with strong ties to i may be likely to vote the same way, while others behave oppositely. Coalitions S and $S \cup i$ which differ only in i's presence will consequently have very different P-probabilities depending on whether S includes i's fellow travelers or opponents. But if the probability P' and value v' of each coalition $T \subseteq N'$ in the 'misspecified' game without i are defined in a probabilistically correct way, i.e., if the misspecified game G' equals G_{-i}, then the assessment of any actor $j \neq i$ should be unaffected by whether one considers G or G_{-i}. Consistency can thus be seen as formalizing robustness to probabilistically correct misspecifications.

Proposition 1. *The prediction value is anonymous, linear, and consistent.*

Proposition 1 is not enough to fully characterize the PV. For instance, for every $a, b \in \mathbb{R}$, the extended value $\varphi_i^{(a,b)}(N, v, P) = a \cdot \mathbb{E}_{P|i}[v(S)] + b \cdot \mathbb{E}_{P|\neg i}[v(S)]$ satisfies anonymity, linearity and consistency.[15] Our characterization in Theorem 1 below will use that if an extended value is linear and consistent, it is fully determined by its image for the subclass of all 2-player probabilistic games.[16] The question then is how the extended values of 2-player probabilistic games should suitably be restricted.

Before giving an answer it is worth recalling two implications of i being part of the formed coalition: first, i's presence means that i contributes to the formed coalition her voting weight, productivity, etc. This reveals information about the expected worth directly. But, second, i's presence also affects the expected worth indirectly because it reveals information about the presence and contributions of other players if the behavior of $N \setminus i$ and of i are not statistically independent. In case of independence, i.e., if the presence of $i \in N$ has *no* informational value according to P, and if moreover i is a null player in the TU-game (N, v), then a reasonable extended value can be expected to assign zero to i. If, in contrast, knowledge of the behavior of null player i does change the odds of a proposal being passed, then i has positive informational value.

For illustration, consider a voting game in which j is a dictator according to the rules formalized by v. Let the voting behavior of j be perfectly correlated with that of some other player i (formally a null player). Now note that it is not part of the model (N, v, P), which mathematically describes the rules of the collective decision body involving i and j and the random *outcomes* of coalition formation processes, *why* the votes of i and j always coincide. 'Null player' i might simply follow 'dictator' j in all his decisions. Alternatively, player i could be irrelevant

[15] To see this, observe that the desired properties are attained by the extended values considered in Lemma 1 (i) and (ii) below, and are preserved by linear combinations.

[16] Lemma 3 below allows to characterize all anonymous, linear and consistent extended values by setting $\varphi_i(\{i, j\}, u_j, P) := f_{ij}(P)$ and $\varphi_i(\{i\}, u_i, P) := g_i(P)$ for anonymous functions f, g. The latter can depend on P in arbitrary anonymous ways. This is why the properties in Proposition 1 do not yield an analogue to Weber's (1988) elegant marginal contributions-based formula for linear positive values which have the dummy property, i.e., probabilistic values.

merely from a formal perspective, i.e., have no say *de jure*; while it is her who imposes all her wishes on j – that is, she rules *de facto*. In either case the informational values of i and j are identical. They are also maximal (and could plausibly be normalized to, say, 1) in the sense that the outcome can be predicted perfectly when knowing that i or j votes *yes* or *no*.

We combine the requirement that an independent null player i should be assigned an extended value of zero with the requirement that i has a value of one in the considered perfect correlation case as follows:[17]

Definition 6. An extended value φ satisfies the *informational dummy-dictator property (IDDP)* if for $i \in N$ and $|N| = 2$

$$\varphi_i(\{i,j\}, u_j, P) = P|i(ij) - P|\neg i(j). \tag{26}$$

Regarding dictators themselves it makes sense to impose the following for 1-player probabilistic games:

Definition 7. An extended value φ satisfies *full control* if $\varphi_i(\{i\}, u_i, P) = 1$ for all i, P where $P(\{i\}) > 0$, and $\varphi_i(\{i\}, u_i, P) = 0$ otherwise.

This formalizes that if N consists of just a single player $i \in \mathbb{N}$ with $v(i) = u_i(i) = 1$ then i's importance or the difference that i makes to this game should plausibly be evaluated as unity. Immediately from the definition of the PV we obtain

Proposition 2. *The prediction value satisfies full control and IDDP .*

Remark 3. IDDP implies a positive extended value for a null player i even if i's behavior is imperfectly but still positively correlated with that of a dictator j. This is, e.g., the case when a *yes*-vote by i is made more likely by most other players voting *yes*, i.e., for the implicit probabilistic model behind the Shapley value. For a probabilistic game with a dictator where P reflects any Shapley value-like probabilistic assumptions, this means that PV and Shapley value ϕ will *not* coincide because the Shapley value of null players is zero.

We have the following characterization result:

Theorem 1. *There is a unique extended value φ which satisfies linearity, consistency, full control and IDDP . It is anonymous and $\varphi \equiv \xi$.*

We note that the four properties in Theorem 1 are independent and non-redundant.

Lemma 1.

(i) The extended value

$$\varphi_i^1(N, v, P) = \sum_{S \subseteq N, i \in S} \alpha_S \cdot \mathbb{E}_{P|i}(u_S), \tag{27}$$

with $v = \sum_{S \subseteq N} \alpha_S \cdot u_S$ being the unique decomposition of v into unanimity games, satisfies linearity, consistency, full control but not IDDP .

(ii) The extended value

$$\varphi_i^2(N, v, P) = \xi_i(N, v, P) - \varphi_i^1(N, v, P) \tag{28}$$

satisfies linearity, consistency, IDDP but not full control.

[17] The case of independence corresponds to $P|i(ij) = P|\neg i(j)$, while the correlated dictator case amounts to $P|i(ij) = 1$ and $P|\neg i(j) = 0$.

(iii) The extended value

$$\varphi_i^3(N,\nu,P) = \sum_{\substack{S\subset N,\\ i\in S,|S|\leqslant 2}} \nu(S)\cdot P|i(S) - \sum_{\substack{T\subset N,\\ i\notin T,|T|\leqslant 2}} \nu(T)\cdot P|\neg i(T) \tag{29}$$

satisfies linearity, full control, IDDP but not consistency.

(iv) Let $|N| \geqslant 3$ *and* $\nu = \sum_{S\subseteq N} \alpha_S \cdot u_S$*. The extended value*

$$\varphi_i^4(N,\nu,P) = \sum_{S\subseteq N,\alpha_S\neq 0} \xi_i(N,u_S,P) \tag{30}$$

satisfies consistency, full control, IDDP but not linearity.

5. Relation between prediction value and probabilistic values

The example values discussed earlier (like $\phi, \beta, \Phi^+, \Phi^-$) all are probabilistic values, i.e., they have in common that they weight marginal contributions of a player by some probability measure. We already noted in Remark 1 that the natural extension of the Banzhaf value agrees with the prediction value if $P(S) \equiv 2^{-|N|}$. We now study the relationship between (extended) probabilistic values and the prediction value somewhat more generally.

Following Weber (1988), the class of probabilistic values is characterized by (i) linearity, (ii) *positivity*: $\varphi(N,\nu,P) \geqslant 0$ if ν is monotonic and (iii) the *dummy player property*: $\varphi_i(N,\nu,P) = \nu(i)$ if i is a dummy. The PV is not a probabilistic value. It is linear (see Proposition 1) but violates (ii) and (iii).[18]

Theorem 2. *Let* Ψ *be a probabilistic value as defined in (6). The identity* $\Psi(\cdot,Q) \equiv \xi(\cdot,P)$ *holds for* $n > 1$ *if and only if there exist probabilities* $0 < \tilde{p}_i < 1$ *for each player such that*

$$Q_i(S\cup i) = P|i(S\cup i) = \prod_{j\in S} \tilde{p}_j \cdot \prod_{j\in N\setminus(S\cup i)} \left(1-\tilde{p}_j\right) \tag{31}$$

holds for all $S \subseteq N\backslash i$*,* $i \in N$ *and*

$$P(S) = \prod_{j\in S} \tilde{p}_j \cdot \prod_{j\in N\setminus S} \left(1-\tilde{p}_j\right) \tag{32}$$

holds for all $S \subseteq N$*.*

The theorem shows that a probabilistic value Ψ is a PV if and only if it is a multinomial value $g^{\tilde{p}}$, i.e., satisfies (11), for an interior vector of probabilities $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_n)$. This also clarifies the connection between semivalues and the prediction value. Namely, a semivalue f^q can be interpreted as a PV if and only if f^q is also a multinomial value $g^{\tilde{p}}$. Identity $f^q \equiv g^{\tilde{p}}$ and the fact that weightings $Q_i(S)$ can depend only on $|S|$ for semivalues imply (i) $\tilde{p}_i = \tilde{p}_j =: p$ for all $i, j \in N$ in Theorem 2 and (ii) q satisfies (10) for p. So we obtain:

[18]See Remark 3 on the possibility of $\xi_i(N,\nu,P) > 0$ for a null player i. Positivity is violated, e.g., for the dummy-dicator setting (26) when i and j always vote contrary such that $P|i(ij) = 0$ and $P|\neg i(j) = 1$.

Corollary 1. *For a given semivalue* f^q *and* $n > 1$ *there exists* P *such that* $f^q(\cdot) \equiv \xi(\cdot, P)$ *on* $\mathcal{G}^N$ *if and only if* f^q *is a binomial semivalue, i.e.,* q *satisfies (10) for some* $p \in (0, 1)$.[19]

Because the Shapley value ϕ is no binomial semivalue and differs from any such value on $\mathcal{G}^N$ when $|N| \geqslant 3$, we have:

Corollary 2. *For* $n \geqslant 3$ *there exists no* P *such that* $\phi(\cdot) \equiv \xi(\cdot, P)$ *on* $\mathcal{G}^N$.

6. Prediction values in the Dutch Parliament 2008–2010

As illustration of the prediction value's practical applicability and of how its informational importance indications can be very different from power ascriptions by traditional values, we consider the seat distribution and voting behavior in the Dutch Parliament between 2008 and 2010. This was the period of the left-centered *Balkenende IV* government, which consisted of Christian democrats from the CDA and Christen Unie parties and the social democratic PvdA.

	CDA	CU	D66	GL	PvdA	PvdD	PVV	SGP	SP	Verdonk	VVD
Seats	41	6	3	7	33	2	9	2	25	1	21
β	0.597	0.073	0.038	0.089	0.398	0.026	0.120	0.026	0.306	0.013	0.200
ϕ	0.317	0.036	0.021	0.044	0.225	0.015	0.061	0.015	0.155	0.007	0.104
Φ^+	0.665	0.040	0.005	0.051	0.283	0.004	0.074	0.004	0.235	0.001	0.210
Φ^-	0.660	0.021	0.004	0.050	0.434	0.005	0.061	0.002	0.140	0.000	0.131
ξ	0.782	0.318	0.248	0.468	0.330	0.023	0.369	0.182	0.217	0.217	0.278

TABLE 1. Values in the Dutch Parliament

The distribution of the 150 seats in parliament between its eleven parties is displayed in the top part of Table 1. The three government parties held a majority of 80 out of 150 seats. When voting on non-constitutional propositions, the Dutch Parliament applies simple majority rule. It is straightforward to define a voting game with this information, and to calculate the corresponding *a priori* Banzhaf and Shapley values β and ϕ.

We used the parliamentary information system *Parlis*[20] in order to extract information on members, meetings, votes and decisions on propositions in the 2008–2010 period. From the records of regular plenary voting rounds, where parties vote as blocks, we derived the empirical frequencies of the 2^{11} conceivable divisions into *yes* and *no*-camps from 2720 observations.[21] Defining P by these empirical frequencies, we calculated the corresponding prediction values ξ_i

[19]In some definitions in the literature the extreme cases $p = 0$ and $p = 1$ are allowed, too, with the convention $0^0 = 1$. For $p = 0$ we would get the *dictatorial index* and for $p = 1$ the *marginal index*. See Owen (1978) for details. However, note that neither $p = 0$ nor $p = 1$ satisfy the conditions from Theorem 2.

[20]The data is available through `http://data.appsvoordemocratie.nl`

[21]We pooled all regular plenary votes in order to illustrate the simplest way in which data can be used to infer interdependencies in a voting body – one might want to split the data with respect to topics, or weight distinct calls by their importance, in actual political analysis. Note that the Dutch Parliament's chairperson assumes that parties vote as blocks unless some MP demands voting by call. Only then can members of the same party vote differently. We excluded such cases of 'non-coherent voting' from our analysis.

of the parties as well as their positive and negative conditional decisiveness values Φ_i^+ and Φ_i^- defined in (14) and (15). A summary of the results is given in the bottom part of Table 1.

The PV-scores ξ_i of Dutch parties tend to be higher than their respective traditional Banzhaf or Shapley power measures β_i and ϕ_i, and even the decisiveness measures Φ_i^+ and Φ_i^- which incorporate the same empirical estimate of P. In particular, the prediction value ascribes rather substantial numbers also to small parties like D66, SGP, or Verdonk.

	CDA	CU	D66	GL	PvdA	PvdD	PVV	SGP	SP	Verdonk	VVD
CDA	1.000	0.267	0.263	0.483	0.237	-0.044	0.324	0.221	-0.026	-0.026	0.012
CU	0.267	1.000	0.631	0.348	0.601	0.015	0.178	0.459	0.094	0.094	0.158
D66	0.263	0.631	1.000	0.348	0.811	0.044	0.169	0.693	0.034	0.034	-0.008
GL	0.483	0.348	0.348	1.000	0.315	-0.003	0.171	0.259	0.019	0.019	0.068
PvdA	0.237	0.601	0.811	0.315	1.000	0.040	0.161	0.714	0.027	0.027	-0.003
PvdD	-0.044	0.015	0.044	-0.003	0.040	1.000	0.198	0.171	0.536	0.536	0.389
PVV	0.324	0.178	0.169	0.171	0.161	0.198	1.000	0.203	0.263	0.263	0.285
SGP	0.221	0.459	0.693	0.259	0.714	0.171	0.203	1.000	0.110	0.110	0.025
SP	-0.026	0.094	0.034	0.019	0.027	0.536	0.263	0.110	1.000	1.000	0.554
Verdonk	-0.026	0.094	0.034	0.019	0.027	0.536	0.263	0.110	1.000	1.000	0.554
VVD	0.012	0.158	-0.008	0.068	-0.003	0.389	0.285	0.025	0.554	0.554	1.000

TABLE 2. Correlation coefficients for 2008–2010 votes in Dutch Parliament

This reflects specificities of the political situation in the Netherlands and that the PV picks up corresponding correlations between the voting behavior of different parties. Varying majorities at calls are quite common in the Dutch Parliament. The member parties of the government do not necessarily vote the same way; some are frequently supported by smaller opposition parties. The correlation coefficients reported in Table 2 indicate, for instance, that SGP and D66 quite commonly voted the same way as CU and PvdA. Their PV numbers hence differ much less than their seat shares.

Verdonk and SP constitute an extreme case in this respect. The former is commonly considered as right-wing, the latter as a left-extremist party; still both voted the same way at each call in the data set (presumably having different reasons). Perfect correlation of their votes implies that both have the same prediction value – despite SP having 25 seats and Verdonk but one: knowing either's vote in advance would have been equally valuable for predictive purposes. Measures based on marginal contributions, in contrast, clearly favor SP over Verdonk (though less so if the *a posteriori* correlation between SP's and Verdonk's votes is ignored). Interestingly, the GL party has the second-highest prediction value: despite it not being in government and having only the sixth-largest seat share, support by GL was a better predictor of a bill's success than support by any except the biggest party (CDA).

7. CONCLUDING REMARKS

Traditional semivalues like the Shapley or Banzhaf values and the prediction value provide two qualitatively distinct perspectives on the importance of the members of a collective decision body. One highlights the difference that an ad-hoc change of a given player i's membership in the coalition which eventually forms would make from an ex ante perspective; the other stresses the difference that the change of a player's presumed membership makes for one's ex ante assessment of realized worth. As the figures in Table 1 illustrate, both can differ widely in case players' behavior exhibits interdependencies. But, as formalized by Theorem 2, they coincide in case of statistical independence. The latter is presumed by the behavioral model underlying, e.g., the Banzhaf value, but incompatible with that underlying the Shapley value. For independent individual voting decisions, the conditioning on different votes of player i adds no behavioral information to the formal fact of i's weight contribution to either the *yes* or *no* camp. Then i's informational importance and i's voting power or influence – reflected by sensitivity of the collective decision to a last-minute change of i's behavior – are aligned.[22]

We note that the prediction value does not distinguish between correlation and causation in cases of interdependence. For illustration, consider decisions by a weighted voting body in which some player i has zero weight but all other players' decisions are perfectly correlated with that of i. Player i's prediction value is then one irrespective of whether (i) players $j \neq i$ 'follow' i and cast their weight as their supreme leader i would if he had any, (ii) $i \neq k$ and all players $j \neq k$ follow a specific other player k, or (iii) all players debate the merit of a proposal based on different initial inclinations and collective opinion dynamics converge to, for instance, the majority inclination.[23] Since knowing i's *decision* – rather than i's initial inclination – will always fully reveal the realized outcome, we regard finding $\xi_i = 1$ in all three scenarios a feature, not a flaw.

However, this example points to an interesting extension of the described "difference of conditional expected values"-approach to measuring importance. Namely, start with a given description (N, v, P) of a decision body where P corresponds to, say, the Banzhaf uniform distribution and augment it by the formal description of a social opinion formation process which defines a mapping from players' binary initial voting inclinations to a distribution over final ones after social interaction. One can then capture a player i's combined social *and* formal influence in the decision body by answering the question: how much does knowing that i's *initial inclination* is in favor (or against) the given proposal modify the final outcome which is to be expected? We conjecture that this approach actually has advantages over extending marginal contribution-based analysis to social interaction,[24] and plan to pursue this extension in future research.

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[22]In the case of the Banzhaf value, coincidence between voter i's influence as picked up by i's average marginal contribution and the informational effect of knowing i's vote has been hinted at by Felsenthal and Machover (1998, 3.2.12–15).

[23]See Grabisch and Rusinowska's (2010) related work on possibilities to aggregate individual influence in command structures.

[24]See, for instance, the power scores derived from swings in societies with opinion leaders by van den Brink et al. (2013).

APPENDIX

Proof of Proposition 1. Anonymity and linearity of ξ are obvious from Definition 1. To prove consistency, consider $(N, v, P) \in \mathcal{PG}$ and let $i \in N$ be dependent in v. Let $j \in N \setminus i$ and $S \subseteq N \setminus ij$. Equations (12) and (24) imply that

$$P_{-i}|j\,(S \cup j) = P|j\,(S \cup j) + P|j\,(S \cup i \cup j) \tag{33}$$

and

$$P_{-i}|\neg j\,(S) = P|\neg j\,(S) + P|\neg j\,(S \cup i)\,. \tag{34}$$

By using the definition of v_{-i} and invoking equality (33) one can verify that

$$P_{-i}|j\,(S \cup j)\, v_{-i}\,(S \cup j) = P|j\,(S \cup i \cup j)\, v\,(S \cup i \cup j) + P|j\,(S \cup j)\, v\,(S \cup j)\,. \tag{35}$$

Similarly, by definition of v_{-i} together with (34), we get

$$P_{-i}|\neg j\,(S)\, v_{-i}\,(S) = P|\neg j\,(S \cup i)\, v\,(S \cup i) + P|\neg j\,(S)\, v\,(S)\,. \tag{36}$$

One can then infer

$$\begin{aligned}
\xi_j(N_{-i}, v_{-i}, P_{-i}) &= \sum_{S \subseteq N \setminus ij} \left\{ P_{-i}|j\,(S \cup j)\, v_{-i}\,(S \cup j) - P_{-i}|\neg j\,(S)\, v_{-i}\,(S) \right\} \\
&= \sum_{S \subseteq N \setminus ij} \Big[\left\{ P|j\,(S \cup i \cup j)\, v\,(S \cup i \cup j) + P|j\,(S \cup j)\, v\,(S \cup j) \right\} \\
&\qquad\qquad - \left\{ P|\neg j\,(S \cup i)\, v\,(S \cup i) + P|\neg j\,(S)\, v\,(S) \right\} \Big] \\
&= \sum_{S \subseteq N \setminus j} \left\{ P|j\,(S \cup j)\, v\,(S \cup j) - P|\neg j\,(S)\, v\,(S) \right\} \\
&= \xi_j(N, v, P),
\end{aligned}$$

where the second equality uses (35) and (36), and the third one follows by shifting the corresponding terms from inside the square brackets to the outer summation.

Proof of Theorem 1. The proof proceeds in three steps. First, in Lemma 2 we prove for $|N| = 2$ that linearity and consistency imply that an extended value is determined by unanimity games. Second, we generalize this to all probabilistic games in Lemma 3. Finally, we show that the full control property and IDDP characterize the PV for 2-player probabilistic games and hence probabilistic games in general.

Lemma 2. *Consider an extended value φ that is linear on the space of all 2-player probabilistic games and consistent. For any set N with $|N| = 2$, the mapping $(N, v, P) \mapsto \varphi(N, v, P)$ is fully determined by the numbers*

$$x_{ij} := \varphi_i(N, u_j, P) \;\; \text{for } i, j \in N. \tag{37}$$

Proof. Let P be a fixed probability distribution on 2^N with $N = \{i, j\}$. The set of unanimity games $\{u_i, u_j, u_{ij}\}$ forms a basis for the space of all TU games on N. In particular, for any $(N, v) \in \mathcal{G}^N$ there are constants $\alpha_i, \alpha_j, \alpha_{ij}$ such that

$$v \equiv \alpha_i u_i + \alpha_j u_j + \alpha_{ij} u_{ij}. \tag{38}$$

And thus, for arbitrary P and $i \in N$, φ's linearity implies

$$\varphi_i(N, v, P) = \alpha_i \underbrace{\varphi_i(N, u_i, P)}_{:=x_{ii}} + \alpha_j \underbrace{\varphi_i(N, u_j, P)}_{:=x_{ij}} + \alpha_{ij} \underbrace{\varphi_i(N, u_{ij}, P)}_{:=x_{i,ij}}. \tag{39}$$

We need to show that $x_{i,ij}$ and $x_{j,ij}$ are fully determined by x_{ii} and x_{ij}.

To see this, notice first that both players are dependent in (N, u_{ij}, P). So we may consider the reduced game obtained by j's removal, which involves $N_{-j} = \{i\}$ and

$$\begin{aligned} &P_{-j}(\varnothing) = P(\varnothing) + P(j), && P_{-j}(i) = P(i) + P(ij), \\ &(u_{ij})_{-j}(\varnothing) = 0, && (u_{ij})_{-j}(i) = \begin{cases} \frac{P(ij)}{P(i)+P(ij)} & \text{if } P(i) + P(ij) > 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{40}$$

In case $P(i) + P(ij) > 0$, we have

$$\begin{aligned} \varphi_i(N, u_{ij}, P) &= \varphi_i\big(\{i\}, \tfrac{P(ij)}{P(i)+P(ij)} \cdot u_i, P_{-j}\big) \\ &= \tfrac{P(ij)}{P(i)+P(ij)} \cdot \varphi_i(\{i\}, u_i, P_{-j}) \\ &= \tfrac{P(ij)}{P(i)+P(ij)} \cdot \varphi_i(N, u_i, P) = \tfrac{P(ij)}{P(i)+P(ij)} \cdot x_{ii}, \end{aligned} \tag{41}$$

where the first equality invokes consistency, the second linearity, and the third one exploits that $(\{i\}, u_i, P_{-j})$ is the reduction of (N, u_i, P) by player j and again consistency. When $P(i) = P(ij) = 0$ we have $\varphi_i(N, u_{ij}, P) = 0$ because in this case $(u_{ij})_{-j}(\{i\}) = 0$ by Definition 4, so that $\big(u_{ij}\big)_{-j}$ is the all-zero game $\mathbf{0}$ in that case. Consistency requires $\varphi_i(N, u_{ij}, P) = \varphi_i(\{i\}, (u_{ij})_{-j}, P_{-j}) = \varphi_i(\{i\}, \mathbf{0}, P_{-j}) = 0$ due to linearity.

In summary,

$$x_{i,ij} = \begin{cases} \frac{P(ij)}{P(i)+P(ij)} \cdot x_{ii} & \text{if } P(i) + P(ij) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{42}$$

And in a similar fashion one obtains

$$x_{j,ij} = \begin{cases} \frac{P(ij)}{P(j)+P(ij)} \cdot x_{jj} & \text{if } P(j) + (Pij) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

□

For any $v \equiv \alpha_i u_i + \alpha_j u_j + \alpha_{ij} u_{ij}$ we have

$$\varphi_i(N, v, P) = \begin{cases} \alpha_j \cdot x_{ij} + \Big(\alpha_i + \frac{\alpha_{ij} \cdot P(ij)}{P(i)+P(ij)}\Big) \cdot x_{ii} & \text{if } P(i) + P(ij) > 0, \\ \alpha_j \cdot x_{ij} + \alpha_i \cdot x_{ii} & \text{otherwise} \end{cases} \tag{44}$$

and an analogous expression for $\varphi_j(N, v, P)$. This finding can be generalized from just two players to arbitrary N:

Lemma 3. *Let φ be a consistent and linear extended value. Then the mapping $(N, v, P) \mapsto \varphi(N, v, P)$ is fully specified by the parameters in* (37).

Proof. Using the n-player unanimity games as a basis for $\mathcal{PG}^N$ one can always write

$$v \equiv \sum_{\varnothing \subsetneq T \subseteq N} \alpha_T u_T. \tag{45}$$

Letting $i \in N$ be an arbitrary but fixed player, we will use induction on n in order to prove the following

Claim: There exist β_{ij}, depending on the α_T and P, such that

$$\varphi_i(N, v, P) = \sum_{j=1}^{n} \beta_{ij} x_{ij} \text{ where } x_{ij} := \varphi_i(N, u_j, P). \tag{46}$$

The claim is obvious for a single player and was proven for $|N| = 2$ in Lemma 2. In view of linearity, it suffices to prove the statement for unanimity games u_T, where nothing needs to be shown when the cardinality of T is one. So we consider $|N| \geqslant 3$, $|T| \geqslant 2$ and assume that the statement is true for all player sets N of cardinality $n-1$. Let $j \in N \setminus i$ be a player, which must be dependent in u_T because $|T| \geqslant 2$. Now we consider the reduced game $(N_{-j}, (u_T)_{-j}, P_{-j})$. From consistency we conclude

$$\varphi_i(N, u_T, P) = \varphi_i(N_{-j}, (u_T)_{-j}, P_{-j}).$$

Applying the induction hypothesis implies the existence of β'_{ik}, which depend on P_{-j} and hence on P, such that

$$\varphi_i(N, u_T, P) = \sum_{k=1, k \neq j}^{n} \beta'_{ik} \varphi_i(N_{-j}, u_k, P_{-j}).$$

Since $(u_k)_{-j} = u_k$ the reduced game of (N, u_k, P) is given by (N_{-j}, u_k, P_{-j}) for all $1 \leqslant k \leqslant n$ with $j \neq k$. Inserting $\varphi_i(N_{-j}, u_k, P_{-j}) = \varphi_i(N, u_k, P) = x_{ik}$ then proves the claim . $\square$

We remark that the coefficients β_{ij} referred to in the above proof get quite complicated for increasing n. In the following we will use only the fact that they are well-defined given v and P.

Proof of Theorem 1. To complete the proof we now show how the values $x_{ii} = \varphi_i(N, u_i, P)$ and $x_{ij} = \varphi_i(N, u_j, P)$ can be computed from the corresponding values for the player set $N' = \{i, j\}$. Since $(u_i)_{-j} = u_i$ for all $i \neq j$ we can recursively conclude from consistency

$$\varphi_i(N, u_i, P) = \varphi_i(\{i, j\}, u_i, P^\star) \text{ and} \tag{47}$$

$$\varphi_i(N, u_j, P) = \varphi_i(\{i, j\}, u_j, P^\star), \tag{48}$$

where

$$P^\star(S) = \sum_{T \subseteq N \setminus ij} P(S \cup T) \text{ for any } S \subseteq \{i, j\}. \tag{49}$$

Using equation (49) and similarly defining

$$P'(S) = \sum_{T \subseteq N \setminus i} P(S \cup T) \text{ for any } S \subseteq \{i\}, \tag{50}$$

we conclude $\varphi_i(\{i\}, u_i, P') = \varphi_i(\{i, j\}, u_i, P^\star)$ from consistency. Thus, the full control property, in connection with consistency and linearity, implies $x_{ii} = 1$ for all player sets N (containing player i). If φ satisfies IDDP the values of x_{ij} are determined, and hence φ is determined on the class of 2-player probabilistic games. Then $\varphi \equiv \xi$ follows from Lemma 3. Finally note that the full control property and IDDP do not depend on the labeling of the players, which implies anonymity.

Proof of Lemma 1.
(i) Linearity is inherited from the expected value $\mathbb{E}_{P|i}$.
To see consistency we first verify that $\mathbb{E}_{P_{-j}|i}\left([u_S]_{-j}\right) = \mathbb{E}_{P|i}(u_S)$ for any probability measure P. If $\sum_{U\subseteq N\setminus j,\, i\in U} P_{-j}(U) = 0$, then both sides of the equation are equal to zero. So, we may assume $\sum_{U\subseteq N\setminus j,\, i\in U} P_{-j}(U) > 0$ in the following. For the reduced game $G_{-j} = (N_{-j}, v_{-j}, P_{-j})$ we have

$$\begin{aligned}
\mathbb{E}_{P_{-j}|i}\left[[u_S]_{-j}\right] &= \sum_{\substack{T\subseteq N\setminus j,\\ i\in T}} P_{-j|i}(T)\cdot [u_S]_{-j}(T) = \sum_{\substack{T\subseteq N\setminus j,\\ i\in T}} \frac{P_{-j}(T)}{\sum_{U\subseteq N\setminus j,\, i\in U} P_{-j}(U)}\cdot [u_S]_{-j}(T)\\
&= \frac{1}{\sum_{U\subseteq N\setminus j,\, i\in U} P_{-j}(U)} \sum_{\substack{T\subseteq N\setminus j,\\ i\in T}} P_{-j}(T)\cdot [u_S]_{-j}(T)\\
&= \frac{1}{\sum_{U\subseteq N,\, i\in U} P(U)} \sum_{\substack{T\subseteq N\setminus j,\\ i\in T}} P_{-j}(T)\cdot [u_S]_{-j}(T)\\
&= \frac{1}{\sum_{U\subseteq N,\, i\in U} P(U)} \sum_{\substack{T\subseteq N\setminus j\\ i\in T}} \{P(T) + P(T\cup j)\}\cdot [u_S]_{-j}(T).
\end{aligned}$$

Inserting (25) provides

$$\begin{aligned}
\mathbb{E}_{P_{-j}|i}\left[[u_S]_{-j}\right] &= \frac{1}{\sum_{U\subseteq N,\, i\in U} P(U)} \sum_{\substack{T\subseteq N\setminus j,\\ i\in T}} \left\{P(T)\cdot u_S(T) + P(T\cup j)\cdot u_S(T\cup j)\right\}\\
&= \frac{1}{\sum_{U\subseteq N,\, i\in U} P(U)} \sum_{i\in T\subseteq N} P(T)\cdot u_S(T) = \mathbb{E}_{P|i}(u_S).
\end{aligned}$$

Note that $v_{-j} = \sum_{S\subseteq N} \alpha_S \cdot [u_S]_{-j}$ and hence

$$\begin{aligned}
\varphi^1_i(N_{-j}, v_{-j}, P_{-j}) &= \sum_{\substack{S\subseteq N\setminus j,\\ i\in S}} \alpha_S \cdot \mathbb{E}_{P_{-j}|i}\left([u_S]_{-j}\right) = \sum_{\substack{S\subseteq N\setminus j,\\ i\in S}} \alpha_S\cdot \mathbb{E}_{P|i}(u_S)\\
&= \varphi^1_i(N, v, P)
\end{aligned}$$

which confirms consistency of φ^1.

Full control follows from $\varphi^1_i(\{i\}, u_i, P) = \mathbb{E}_{P|i}[u_i(S)] = P|i(\{i\})$. This is 1 if $P(\{i\}) > 0$ and otherwise 0 by the definition of $P|i$.

In remains to be shown that φ^1 does not satisfy IDDP . For the unanimity game u_j note that $u_j = \sum_{S\subseteq N} \alpha_S \cdot u_S$ provides

$$\alpha_S = \begin{cases} 1 & \text{for } S = \{j\},\\ 0 & \text{otherwise.}\end{cases}$$

However, the summation in (27) is over all S with $i \in S$ such that

$$\varphi^1_i(\{i,j\}, u_j, P) = 0 \tag{51}$$

in contradiction to (26).

(ii) φ^2 inherits linearity and consistency from ξ and φ^1. Inserting (51) into (28) provides

$$\varphi_i^2(\{i,j\},u_j,P)=\xi_i(\{i,j\},u_j,P)$$

such that φ_i^2 inherits IDDP from ξ_i. As both ξ and φ^1 satisfy full control we get

$$\varphi_i^2(\{i\},u_i,P)=0,$$

in contradiction to Definition 7.

(iii) Linearity is obvious. For $|N|\leqslant 2$ the extended value φ^3 is identical to the PV and the latter satisfies full control and IDDP . For a counterexample to consistency consider a game $G_{-j}=(N,v,P)$ with $|N|=3$ and perfect correlation $P(N)=1/2=P(\varnothing)$. Here,

$$\varphi_i^3(N,v,P)=0 \text{ for all } i\in N. \tag{52}$$

However, for the reduced game $G_{-j}=(N_{-j},v_{-j},P_{-j})$ we get

$$\begin{aligned}
N_{-j} &= N\setminus j,\\
P_{-j}(S) &= P(S)+P(S\cup j) \text{ for all } S\subseteq N\setminus j\\
&= \begin{cases} 1/2 & \text{for } S\in\{N\backslash j,\varnothing\},\\ 0 & \text{otherwise,}\end{cases}\\
v_{-j}(S) &= \begin{cases} v(N) & \text{for } S=N\backslash j,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

For φ^3 follows

$$\varphi_i^3(N_{-j},v_{-j},P_{-j})=v(N)-v(\varnothing)=v(N) \text{ for all } i\in N$$

which contradicts (52).

(iv) Consider the reduced game $G_{-j}=(N_{-j},v_{-j},P_{-j})$. PV is consistent and therefore

$$\xi_i(N_{-j},(u_S)_{-j},P_{-j})=\xi_i(N,u_S,P) \text{ for all } i\in N\setminus j.$$

We conclude

$$\begin{aligned}
\varphi_i^4(N_{-j},v_{-j},P_{-j}) &= \sum_{S\subseteq N,\alpha_S\neq 0}\xi_i(N_{-j},(u_S)_{-j},P_{-j})\\
&= \sum_{S\subseteq N,\alpha_S\neq 0}\xi_i(N,u_S,P)=\varphi_i^4(N,v,P) \text{ for all } i\in N\setminus j
\end{aligned}$$

which confirms consistency of φ_i^4.
Full control and IDDP follows from $\varphi_i^4(\{i\},u_i,P)=\xi_i(\{i\},u_i,P)$ and $\varphi_i^4(\{i,j\},u_j,P)=\xi_i(\{i,j\},u_j,P)$.
To verify that φ_i^4 is not linear put $w=\sum_{S\subseteq N}\beta_S\cdot u_S$.

$$\varphi_i^4(N,v+w,P)=\sum_{S\subseteq N,\alpha_S+\beta_S\neq 0}\xi_i(N,u_S,P)$$

which is in general not equal to

$$\sum_{S\subseteq N,\alpha_S\neq 0} \xi_i(N,u_S,P) + \sum_{S\subseteq N,\beta_S\neq 0} \xi_i(N,u_S,P).$$

Proof of Theorem 2. The proof is based on three insights, stated in Lemmas 4–6.

Lemma 4. *From* $\Psi(\cdot,Q)\equiv\xi(\cdot,P)$ *follows* $Q_i(S)=P|i(S)$ *for all* $\{i\}\subseteq S\subseteq N$.

Proof. For an arbitrary subset $\{i\}\subseteq S\subseteq N$ we consider the unanimity game u_S and obtain the formulas

$$\xi_i(u_S,P)=\sum_{T\ni i} u_S(T)\cdot P|i(T)-\sum_{T\not\ni i} u_S(T)\cdot P|\neg i(T)=\sum_{T:S\subseteq T} P|i(T)$$

and

$$\Psi_i(u_S,Q)=\sum_{\{i\}\subseteq T\subseteq N} Q_i(T)\Big[u_S(T)-u_S(T\backslash i)\Big]=\sum_{T:S\subseteq T} Q_i(T).$$

Now we prove the proposed statement by induction on the subsets S in decreasing order of their cardinalities using the assumption $\xi_i(u_S,P)=\Psi_i(u_S,Q)$. For the induction start $S=N$ we have $P|i(N)=Q_i(N)$. Using the induction hypothesis for all $S'\subseteq N$ with $|S'|>|S|$ yields $P|i(S)=Q_i(S)$. □

Lemma 5. *From* $\Psi(\cdot,Q)\equiv\xi(\cdot,P)$ *follows* $P|i(U)=P|\neg i(U\backslash i)$ *for all* $\{i\}\subseteq U\subseteq N$ *with* $|U|\geqslant 2$.

Proof. We set $U=N\backslash S\cup i$ so that we have to prove $P|i(N\backslash S\cup i)=P|\neg i(N\backslash S)$ for all subsets $\{i\}\subseteq S\subsetneq N$.

For fixed S we consider the unanimity game $u_{N\backslash S}$ and obtain the formulas

$$\begin{aligned}\xi_i(u_{N\backslash S},P) &= \sum_{T\ni i} u_{N\backslash S}(T)\cdot P|i(T)-\sum_{T\not\ni i} u_{N\backslash S}(T)\cdot P|\neg i(T)\\ &= \sum_{T:\,N\backslash S\subseteq T\subseteq N\backslash\{i\}} \Big(P|i(T\cup i)-P|\neg i(T)\Big)\end{aligned}$$

and

$$\Psi_i(u_{N\backslash S},Q)=\sum_{T\ni i} Q_i(T)\Big[u_{N\backslash S}(T)-u_{N\backslash S}(T\backslash i)\Big]=0.$$

Now we prove the proposed statement by induction on the subsets S in increasing order of their cardinalities using the assumption $\xi_i(u_S,P)=\Psi(u_S,Q)$. For the induction start $S=\{i\}$ we have $P|i(N)-P|\neg i(N\backslash i)=0$, which is equivalent to $P|i(N)=P|\neg i(N\backslash\{i\})$. Using the induction hypothesis for all $S'\subseteq N$ with $|S'|<|S|$ yields $P|i(N\backslash S\cup i)=P|\neg i(N\backslash S)$. □

Put $p_i:=\sum_{T\ni i} P(T)\in[0,1]$ for all $i\in N$. Whenever $p_i>0$ we have $P|i(S)=\frac{P(S)}{p_i}$ for all $\{i\}\subseteq S\subseteq N$ and $P|i(S)=0$ in all other cases. The next lemma excludes the case $p_i=1$ for at least two players.

Lemma 6. *If* $\Psi(\cdot,Q)\equiv\xi(\cdot,P)$ *and if there exists an index* $i\in N$ *with* $p_i=1$*, then* $n=1$.

Proof. From $p_i = \sum_{T \ni i} P(T) = 1$ we conclude $P(S) = 0$ for all $S \subseteq N \backslash i$. Thus we have $P|i(T) = 0$ for all $T \ni i$ with $|T| \geqslant 2$ due to Lemma 5. This yields $P(\{i\}) = Q_i(\{i\}) = 1$ and $Q_j(S) = 0$ for all $S \ni j$, where $(S, j) \neq (\{i\}, i)$, and all $j \in N$ due to Lemma 4. For each $j \in N \backslash i$ we then have $\sum_{S \ni j} Q_j(S) = 0 \neq 1$ – a contradiction. $\square$

Proof of Theorem 2. From Lemma 6 we conclude

$$0 \leqslant p_i := \sum_{T \ni i} P(T) < 1$$

for all $i \in N$. If $p_i = 0$ for an index $i \in N$, then we have $Q_i(S) = 0$ due to Lemma 4, which contradicts the definition of the $Q_i(S)$. Thus we have $0 < p_i < 1$. Later on it will turn out that indeed we can choose $\tilde{p}_i = p_i$.

We have

$$P(S) = \frac{p_i}{1 - p_i} \cdot P(S \backslash i)$$

for all $S \ni i$ with $|S| \geqslant 2$ due to Lemma 5 and $p_i > 0$. Thus inductively we obtain

$$P(S) = \prod_{j \in S \backslash i} \frac{p_j}{1 - p_j} \cdot P(\{i\})$$

for all $i \in N$ and all subsets $S \ni i$ of N.

Inserting the previous equations into $p_i = \sum\limits_{S \ni i} P(S)$ yields

$$p_i = P(\{i\}) \cdot \sum_{S \not\ni i} \prod_{j \in S} \frac{p_j}{1 - p_j} = P(\{i\}) \cdot \prod_{j \in N \backslash i} \left(\frac{p_j}{1 - p_j} + 1 \right) = P(\{i\}) \cdot \prod_{j \in N \backslash i} \frac{1}{1 - p_j}.$$

Thus we have

$$P(\{i\}) = p_i \cdot \prod_{j \in N \backslash i} (1 - p_j),$$

which then yields

$$P(S) = \prod_{j \in S} p_j \cdot \prod_{j \in N \backslash S} (1 - p_j) \tag{53}$$

for all $\varnothing \neq S \subseteq N$. By using $\sum_{S \subseteq N} P(S) = 1$ we conclude that equation (53) is also valid for the empty set and thus for all subsets of N.

Lemma 4 and a short calculation gives also the first formula of the proposed statement.

To verify that the converse holds as well let $0 < p_i < 1$ be given for all $i \in N$ and define

$$P(S) = \prod_{j \in S} p_j \cdot \prod_{j \in N \backslash S} \left(1 - p_j\right),$$

i.e., P is a product measure. Next set

$$Q_i(S \cup i) = P|i(S \cup i) = \prod_{j \in S} p_j \cdot \prod_{j \in N \backslash (S \cup i)} \left(1 - p_j\right),$$

for $i \in N\backslash S$, i.e. the $Q_i(S \cup i)$ derive from the same product measure. We can easily verify $P|i(S) = P|\neg i(S\backslash i)$ for all $S \ni i$ and all $i \in N$. Inserting this into the definition of the prediction value provides $\xi(\cdot, P) = \Psi(\cdot, Q)$.

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