THE NUCLEOLUS OF LARGE MAJORITY GAMES

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ABSTRACT. Members of a shareholder meeting or legislative committee have greater or smaller voting power than meets the eye if the nucleolus of the induced majority game differs from the voting weight distribution. We establish a new sufficient condition for the weight and power distributions to be equal; and we characterize the limit behavior of the nucleolus in case all relative weights become small.

Keywords: nucleolus; power measurement; weighted voting games
JEL classification: C61, C71

1. INTRODUCTION

Among all individually rational and efficient payoff vectors in a game v with transferable utility, the *nucleolus* selects a particularly stable one. It quantifies each coalition's dissatisfaction with a proposed vector x as the gap between the coalition's worth $v(S)$ and the surplus share $\sum_{i\in S} x_i$ that is allocated to members of $S \subseteq N$; then it selects the allocation x^* which involves lexicographically minimal dissatisfaction. In contrast to other prominent point solutions in cooperative game theory, such as the Shapley value, x^* is guaranteed to lie in the *core* of game (N, v) whenever that is non-empty.

Even before the final version of Schmeidler's article which established the definition, existence, uniqueness, and continuity of the nucleolus was published in 1969, Peleg (1968) had applied it to *weighted majority games (WMG)*. In these games the worth of a coalition S of players is either 1 or 0, i.e., S is either winning or losing, and there exists a non-negative quota-and-weights *representation* $[q; w_1, \ldots, w_n]$ such that $v(S) = 1$ iff $\sum_{i\in S} w_i \geq q$. The weight vectors that constitute a representation of a given WMG v for some quota q form a non-singleton convex set $R(v)$.

Peleg highlighted a property of constant-sum WMGs with a *homogeneous* representation, i.e., one where total weight of any minimal winning coalition equals

q: the nucleolus x^* of such a WMG v is contained in $R(v)$, i.e., it is also a representation.[1] Despite this early start, the relation between voting weights and the nucleolus of weighted majority games – constant-sum or not, homogeneous or inhomogeneous – has to the best of our knowledge not been studied systematically so far. This paper is a first attempt to fill this gap.

Discrepancies between weights and the nucleolus matter because the nucleolus is an important indicator of influence in collective decision bodies. It emerges as an equilibrium price vector in models that evaluate voters' attractiveness to competing lobbying groups (see Young 1978; Shubik and Young 1978); more recent theoretical work by Montero (2005, 2006) has established it as a focal equilibrium prediction for strategic bargaining games with a majority rule.[2] So large differences between a voter i's weight w_i and nucleolus x_i^* can mean that the real power distribution in a decision body such as a shareholder meeting is hidden from the casual observer. This intransparency can be particularly problematic for political decision bodies, where voting weight arrangements affect the institution's legitimacy.[3]

This paper investigates absolute and relative differences between players' relative voting weights as defined by vote shares in an assembly, electoral college, etc. and the nucleolus of the implied WMG. We determine an upper bound on their $\|\cdot\|_1$-distance which depends only on quota and maximum weight in a given representation in Lemma 1. The lemma allows to conclude that if the relative weight of every individual voter in player set $\{1,\ldots,n\}$ tends to zero, then the ratio x_i^*/x_j^* of two nucleolus components converges to w_i/w_j for all regular voters i and j as $n \to \infty$ (Prop. 1). This complements analogous limit results in the literature on the Shapley value, the Banzhaf value and voter pivotality on intervals (see Neyman 1982; Lindner and Machover 2004; Kurz et al. 2013) as well as for stationary equilibrium payoffs in legislative bargaining games à la Baron-Ferejohn (see Snyder et al. 2005). We also establish a new sufficient condition for the nucleolus to

[1] A WMG (N,v) is called *constant-sum* if for any $S \subseteq N$ either $v(S)=1$ or $v(N \setminus S)=1$. $S \subseteq N$ is a *minimal winning coalition (MWC)* if $v(S)=1$ and $v(T)=0$ for any $T \subset S$.

[2] Corresponding experimental lab evidence is mixed; see Montero et al. (2008). Non-cooperative foundations of the nucleolus for other than majority games have been given, e.g., by Potters and Tijs (1992) and Serrano (1993, Serrano (1995).

[3] See Le Breton et al. (2012) for nucleolus-based power analysis of the European Union's Council; an early-day weight arrangement meant that Luxembourg had a relative voting weight of 1/17 but zero voting power. – In general, the power-to-weight ratio can differ arbitrarily from 1. For instance, the nucleolus of the WMG with representation $[0.5; (1-\varepsilon)/2, (1-\varepsilon)/2, \varepsilon]$ is $x^* = (1/3, 1/3, 1/3)$ for any $\varepsilon \in (0; 0.5)$.

coincide with given relative weights (Prop. 2). It implies that a finite number of replications brings about full coincidence for any given WMG.

2. NUCLEOLUS

Consider a WMG (N, v) with representation $[q; w_1, \ldots, w_n]$. Using notation $x(S) = \sum_{i \in S} x_i$, a vector $x \in \mathbb{R}^n$ with $x_i \geq v(\{i\})$ and $x(N) = v(N)$ is called an *imputation*. For any coalition $S \subseteq N$ and imputation x, call $e(S, x) = v(S) - x(S)$ the *excess* of S at x. It can be interpreted as quantifying the coalition's dissatisfaction and potential opposition to an agreement on allocation x. For any fixed x let $S_1, \ldots, S_{2^n}$ be an ordering of all coalitions such that the excesses at x are weakly decreasing, and denote these ordered excesses by $E(x) = \big(e(S_k, x)\big)_{k=1,\ldots,2^n}$. Imputation x is *lexicographically less* than imputation y if $E_k(x) < E_k(y)$ for the smallest component k with $E_k(x) \neq E_k(y)$. The *nucleolus* of (N, v) is then uniquely defined as the lexicographically minimal imputation.[4]

As an example, consider (N, v) with representation $[q; w] = [8; 6, 4, 3, 2]$. The nucleolus can be computed as $x^* = (2/5, 1/5, 1/5, 1/5)$ by solving a sequence of linear programs – or by appealing to the sufficient condition of Peleg (1968) after noting that the game is constant-sum and permits a homogeneous representation $[q'; w'] = [3; 2, 1, 1, 1]$. Denoting the *normalization* of weight vector w by $\bar{w}$, i.e., $\bar{w} = w / \sum w_i$, the respective total differences between relative weights and the nucleolus are $\|\bar{w} - x^*\|_1 = 2/15$ for the first and $\|\bar{w}' - x^*\|_1 = 0$ for the second representation (with $\|x\|_1 = \sum |x_i|$).

3. RESULTS

Saying that representation $[q; w]$ is *normalized* if $w = \bar{w}$, we have:[5]

[4]Schmeidler's (1969) original definition did not restrict the considered vectors to be imputations but is usually specialized this way. The set of imputations that minimize just the largest excess, $E_1(x)$, is called the *nucleus* of (N, v) by Montero (2006). Our results are stated for the nucleolus but apply to every element of the nucleus: both coincide under the conditions of Prop. 2; Lemma 1 and Prop. 1 generalize straightforwardly.

[5]All proofs are provided in the Mathematical Appendix.

Lemma 1. *Consider a normalized representation* $[q; w]$ *with* $0 < q < 1$ *and* $w_1 \geq \cdots \geq w_n \geq 0$ *and let* x^* *be the nucleolus of this WMG. Then*

$$\|x^* - w\|_1 \leq \frac{2w_1}{\min\{q, 1-q\}}. \tag{1}$$

If we consider a sequence $\{(\{1, \ldots, n\}, v^{(n)})\}_{n\in\mathbb{N}}$ of n-player WMGs with representations $[q^{(n)}; w^{(n)}]$ such that the normalized quota $\bar{q}^{(n)}$ is bounded away from 0 and 1 (or, more generally, 0 and 1 are no cluster points of $\{\bar{q}^{(n)}\}_{n\in\mathbb{N}}$), and each player i's normalized weight $\bar{w}_i^{(n)}$ vanishes as $n \to \infty$ then Lemma 1 implies

$$\lim_{n\to\infty} \|x^{*(n)} - \bar{w}^{(n)}\|_1 \to 0. \tag{2}$$

Convergence to zero of the total difference between nucleolus components $x_i^{*(n)}$ and relative voting weights $\bar{w}_i^{(n)}$ does not yet guarantee that the nucleolus is asymptotically proportional to the weight vector, i.e., that each ratio $x_i^{*(n)}/x_j^{*(n)}$ converges to w_i/w_j. This can be seen, e.g., by considering

$$\left[q^{(n)}; w^{(n)}\right] = \left[\frac{2n-1}{2}; 1, \underbrace{2, \ldots, 2}_{n-1}\right]. \tag{3}$$

The nucleolus either equals $\left(0, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)$ or $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ depending on whether n is even or odd; ratio $x_1^{*(n)}/x_2^{*(n)} \neq \frac{1}{2}$ alternates between 0 and 1.

But such pathologies are ruled out for players i and j whose weights are "non-singular" in the weight sequence $\left\{w^{(n)}\right\}_{n\in\mathbb{N}}$. Specifically, denote the total number of players $i \in \{1, \ldots, n\}$ with an identical weight of $w_i^{(n)} = \omega$ by $m_\omega(n)$. We say that a player j with weight w_j is *regular* if $m_{w_j}(n) \cdot \bar{w}_j^{(n)}$ is bounded away from 0 by some constant $\varepsilon > 0$. Lemma 1 then implies:[6]

Proposition 1. *Consider a sequence* $\left\{\left[q^{(n)}; (w_1, \ldots, w_n)\right]\right\}_{n\in\mathbb{N}}$ *with corresponding normalized quotas that exclude 0 and 1 as cluster points and with normalized weights satisfying* $\bar{w}_k^{(n)} \downarrow 0$ *for every* $k \in \mathbb{N}$ *as* $n \to \infty$*. Then the nucleolus* $x^{*(n)}$ *of*

[6]We assume $w_j^{(n)} = w_j$ in our exposition. Adaptations to cases where $q^{(n)}$ *and* $w_j^{(n)}$ vary in n are straightforward. The essential regularity requirement is that a voter type's aggregate relative weight does not vanish.

the WMG represented by $\left[q^{(n)}; (w_1, \ldots, w_n)\right]$ *satisfies*

$$\lim_{n\to\infty} \frac{x_i^{*(n)}}{x_j^{*(n)}} = \frac{w_i}{w_j} \tag{4}$$

for any regular players i *and* j.

For a considerable class of games, asymptotic equality of nucleolus and normalized weights can be strengthened to actual equality.[7] Namely, for a fixed n-player WMG with representation $[q; w_1, \ldots, w_n]$ let m_ω denote the number of players that have weight ω; so

$$m^\circ = \min_{i\in\{1,\ldots,n\}} m_{w_i} \geq 1 \tag{5}$$

is the number of occurrences of the rarest weight in vector $w = (w_1, \ldots, w_n)$.

Proposition 2. *Consider a WMG representation* $[q; w]$ *with integer weights* $w_1 \geq \ldots \geq w_n \geq 0$ *and normalization* $[\bar{q}; \bar{w}]$. *Denoting the number of numerically distinct values in* w *by* $1 \leq t \leq n$, *the nucleolus* x^* *of this WMG satisfies*

$$x^* = \bar{w} \;\; \textit{if} \;\; \min\{\bar{q}, 1-\bar{q}\} \cdot m^\circ > 2t{w_1}^2. \tag{6}$$

The proposition refers to *integer* weights. Even though it is not difficult to obtain an integer representation from any given $[q; w]$ with non-integer values, this is an important restriction. In particular, it is not possible to rescale a given weight vector w so as to make the maximal weight w_1 arbitrarily small.[8]

The right-hand side of the inequality in condition (6) is smaller, the smaller the number of different weights in the representation, and the smaller the involved integers (particularly w_1). Similarly, the left-hand side is larger, the greater the number of occurrences of the rarest weight. It follows that condition (6) is most easily met

[7] Non-null players have a positive nucleolus value in this case – in contrast to WMGs in general. So we implicitly establish a sufficient condition for $w_i > 0 \Rightarrow x_i^* > 0$.

[8] Note also that inequality (6) must be violated if two interchangeable players of (N, v) have different weights because $x^* = \bar{w}$ would then contradict the symmetry property of the nucleolus. So as a subtle implication of the integer requirement, weight changes which would destroy a given symmetric or 'type-preserving' representation and satisfy (6) are impossible. Another way to look at this is to begin with a WMG's representation where $w_i \neq w_j$ for interchangeable players i and j and then to replicate all players and weights: after enough replications the two players (types) i and j must lose their interchangeability.

when null players (where $x_i^* = 0$ is known) are removed from the WMG in question and a *minimal integer representation* is considered.[9] This is automatically also a homogeneous representation if any exists.

Our sufficient condition for $x^* = \bar{w}$ is, however, independent of the known homogeneity-based one. The normalization of weights in $[3; 2, 1, 1, 1]$ must, according to Peleg (1968), coincide with the WMG's nucleolus because the game is constant-sum; but our condition (6) is violated. In contrast, the representation $[q; w] = [1500; 4, \ldots, 4, 3, \ldots, 3, 2, \ldots, 2]$ of a 900-player WMG where each of the $t = 3$ weight types occurs $m^\circ = 300$ times satisfies our condition. Hence $x^* = \bar{w}$. Since the game is inhomogeneous,[10] Peleg's finding does not apply.

The left-hand side in condition (6) equals at most half the number of occurrences of the rarest weight, m°, and the right-hand side is bounded below by 2. This, first, implies that the condition cannot be met by WMGs where only one instance of some weight type is involved. This limits Prop. 2's applicability for small-scale games such as $[3; 2, 1, 1, 1]$. But, second, it means that if we consider *ρ-replicas* of any given n-player WMG with integer representation $[q; w]$, i.e., a WMG with quota ρq and ρ instances of any of the n voters in $[q; w]$, then one can compute an explicit number $\tilde{\rho}$ from (6) such that the nucleolus of the resulting ρn-player WMG must coincide with the corresponding normalized weight vector for every $\rho \geq \tilde{\rho}$.[11] This observation echoes the coincidence result obtained by Snyder et al. (2005) for WMG replicas under Baron-Ferejohn bargaining:[12] at least in sufficiently large majority games, voting weight and power are the same.

[9]Uniqueness and other properties of minimal integer representations of WMG are investigated by Kurz (2012).

[10]Coalitions with (a) 300, 100, and 0, (b) 300, 0, and 150, or (c) 300, 1 and 149 players of weights 4, 3, and 2 are minimal winning, and cannot be made to have identical aggregate weights in any representation $[q'; w']$.

[11]For simple majority games which involve equal numbers of voters with weight $\omega = 4$, 3, and 2, condition (6) calls for $m^\circ > 192$. But $x^* = \bar{w}$ already holds after one replication of $[5; 4, 3, 2]$, i.e., $\rho \geq 2$. So tighter bounds might be obtained by different techniques than ours. However, surprising sensitivity of x^* to the game at hand cautions against too high expectations. For instance, $x^* = \bar{w}$ if $w_1 = \ldots = w_5 = 4$ and either $w_6 = \ldots = w_{11} = 1$ or $w_6 = \ldots = w_{13} = 1$ with $\bar{q} = 58\%$; in contrast, $x^* = (1/5, \ldots, 1/5, 0, \ldots, 0)$ if $w_6 = \ldots = w_{12} = 1$. We thank Maria Montero for suggesting this example.

[12]We thank an anonymous referee for pointing out to us that Snyder et al.'s Prop. 2 is in fact a corollary to our Prop. 2, the uniqueness of SSPE payoffs recently established by Eraslan and McLennan (2013), and Montero's (2006) Prop. 7.

MATHEMATICAL APPENDIX

Proof of Lemma 1. Define $w(S) = \sum_{i \in S} w_i$ and $x^*(S) = \sum_{i \in S} x_i^*$. Let $S^+ = \{i \in N \mid x_i^* > w_i\}$ and $S^- = \{i \in N \mid x_i^* \leq w_i\}$. We have $w(S^-) > 0$ because weights cannot exceed nucleolus values for *all* $i \in N$ given $w(N) = x^*(N) = 1$. If $w(S^-) = 1$ then $\|x^* - w\|_1 = 0$.

So let $0 < w(S^-) < 1$ and define $0 \leq \delta \leq 1$ by $x^*(S^-) = (1-\delta)w(S^-)$. We have $x^*(S^+) = w(S^+) + \delta w(S^-)$ and the respective substitutions in decomposition $\|x^* - w\|_1 = \sum_{i \in S^+}(x_i^* - w_i) + \sum_{j \in S^-}(w_j - x_j^*)$ yield

$$\|x^* - w\|_1 = 2\delta w(S^-). \tag{7}$$

Let T be a MWC which is generated by starting with $S = \varnothing$ and successively adding a remaining player i with minimal $x_i^*/\bar{w}_i$ until $w(T) \geq q$.

In case $w(S^-) \geq q$ we then have $x^*(T)/w(T) \leq x^*(S^-)/w(S^-) = 1 - \delta$. Multiplying by $w(T)$, using $q \leq w(T) \leq q + w_1$ and finally $\delta \leq 1$ yields

$$x^*(T) \leq (1-\delta)w(T) \leq (1-\delta)(q + w_1) \leq q(1-\delta) + w_1. \tag{8}$$

This and $q \leq w(T) \leq x^*(T)$ yield $\delta \leq w_1/q$. Applying this and $w(S^-) < 1$ in equation (7) gives $\|x^* - w\|_1 \leq \frac{2w_1}{q}$.

In case $w(S^-) < q$, note that moving from S^- to T involves the weight addition $w(T) - w(S^-)$ which comes with a nucleolus per weight unit of at most $x^*(S^+)/w(S^+)$. So

$$\begin{aligned} x^*(T) &= x^*(S^-) + x^*(T \backslash S^-) \\ &\leq (1-\delta)w(S^-) + \frac{x^*(S^+)}{w(S^+)} \cdot \big(w(T) - w(S^-)\big) \\ &\leq (1-\delta)w(S^-) + \frac{x^*(S^+)}{w(S^+)} \cdot \big(q - w(S^-) + w_1\big) \qquad (9) \end{aligned}$$

where the last inequality uses $w(T) \leq q + w_1$. Rearranging with $x^*(S^+) = w(S^+) + \delta w(S^-)$ and $w(S^-) = 1 - w(S^+)$ yields

$$x^*(T) \leq q + \frac{w(S^+) + \delta w(S^-)}{w(S^+)} \cdot w_1 - \frac{(1-q)\delta w(S^-)}{w(S^+)}. \tag{10}$$

Since $\delta \leq 1$ the right hand side of (10) is at most $q + \big(w_1 - (1-q)\delta w(S^-)\big)/w(S^+)$. So $q \leq x^*(T)$ implies $(1-q)\delta w(S^-) \leq w_1$. Hence $\|x - w\|_1 \leq \frac{2w_1}{1-q}$. $\square$

Proof of Proposition 1. If $x_i^{*(n)} / \bar{w}_i^{(n)} \geq 1+\delta$ or $x_i^{*(n)} / \bar{w}_i^{(n)} \leq 1-\delta$ then $\|x^{*(n)} - \bar{w}^{(n)}\|_1 \geq \delta \cdot m_{w_i}(n) \cdot \bar{w}_i^{(n)} \geq \delta\varepsilon$ for some $\varepsilon > 0$ if i is regular. But $\lim\limits_{n\to\infty} ||x^{*(n)} - \bar{w}^{(n)}||_1 = 0$ (Lemma 1). So $\lim\limits_{n\to\infty} x_i^{*(n)} / \bar{w}_i^{(n)} = 1$ and hence

$$1 = \lim_{n\to\infty} \frac{x_i^{*(n)}}{\bar{w}_i^{(n)}} \cdot \frac{\bar{w}_j^{(n)}}{x_j^{*(n)}} = \lim_{n\to\infty} \frac{x_i^{*(n)}}{x_j^{*(n)}} \cdot \frac{w_j}{w_i} \quad \text{if } i \text{ and } j \text{ are regular.} \tag{11}$$

□

Proof of Proposition 2. It suffices to prove the result in case $w_n > 0$ because $w_i = 0$ directly implies $x_i^* = 0$. We may also assume $0 < \bar{q} < 1$. For each $k \in \{1, \ldots, t\}$ let ω_k denote the normalized weight of a voter i with type k (i.e., $\bar{w}_i = \omega_k$) and, with slight abuse of notation, let x_k^* be this voter/type's nucleolus. Define $r_k = x_k^*/\omega_k$ and w.l.o.g. assume $r_1 \geq \ldots \geq r_t$. Let a denote the largest index such that $r_1 = r_a$ and b be the smallest such that $r_b = r_t$. The claim is true if $a \geq b$. So we suppose $a < b$ and establish a contradiction by showing that we can construct an imputation x^{**} with maximum excess $E_1(x^{**})$ smaller than $E_1(x^*)$.

Writing $\varepsilon = \frac{1}{2} \cdot \min\{\bar{q}, 1-\bar{q}\}$ and $n_k = m_{\omega_k}$, the premise and $t, w_1 \geq 1$ imply

$$w_1 \leq t{w_1}^2 < \varepsilon m^\circ \leq \varepsilon n_k \tag{12}$$

for each $k \in \{1, \ldots, t\}$. Considering ω_k-weighted sums of (12) we obtain

$$\text{(I)} \quad \sum_{k\leq a} w_1\omega_k < \varepsilon \sum_{k\leq a} n_k\omega_k \quad \text{and} \quad \text{(II)} \quad \sum_{k\geq b} w_1\omega_k < \varepsilon \sum_{k\geq b} n_k\omega_k.$$

Moreover, we have

$$\text{(III)} \quad \sum_{k<b} w_1\omega_k < \varepsilon \sum_{k\geq b} n_k\omega_k \quad \text{and} \quad \text{(IV)} \quad \bar{w}_1 + \sum_{k>a} w_1\omega_k < \varepsilon \sum_{k\leq a} n_k\omega_k.$$

Inequality (III) follows from

$$\sum_{k<b} w_1\omega_k < t w_1 \bar{w}_1 = \frac{t{w_1}^2}{w(N)} < \frac{\varepsilon m^\circ}{w(N)} \leq \varepsilon \sum_{k\geq b} n_k\omega_k \tag{13}$$

using $1/w(N) \leq \omega_k \leq \bar{w}_1$ and (12). Similarly, (IV) follows from

$$\bar{w}_1 + \sum_{k>a} w_1\omega_k \leq \bar{w}_1 + (t-1)w_1\bar{w}_1 \leq t\frac{{w_1}^2}{w(N)} < \varepsilon m^\circ \frac{1}{w(N)} \leq \varepsilon \sum_{k\leq a} n_k\omega_k. \tag{14}$$

Let n_k^T denote the number of k-type voters in a coalition $T \subseteq N$ and define

$$D(T) = \frac{\sum\limits_{k \le a} n_k^T \omega_k}{\sum\limits_{k \le a} n_k \omega_k} \quad \text{and} \quad I(T) = \frac{\sum\limits_{k \ge b} n_k^T \omega_k}{\sum\limits_{k \ge b} n_k \omega_k}. \tag{15}$$

$D(T)$ is the share of the total weight of the a "most over-represented" types (all having maximal nucleolus-to-relative weight ratio x_1^*/ω_1) which they contribute in coalition T. Similarly, $I(T)$ is the respective share for the $t-b+1$ "most under-represented" types.

Given a suitably large coalition $S \subseteq N$, replacing w_h members of type k – all with absolute weight w_k – by w_k players of type h yields a coalition S' with $w(S') = w(S)$. But if $r_k > r_h$, such replacement yields $x^*(S') < x^*(S)$. Thus, for a MWC T with *maximum* excess at x^*, i.e., with excess $1 - x^*(T) \ge v(S) - x^*(S)$ for all $S \subseteq N$, there are no k, h with $r_k > r_h$ such that (i) w_1 or more type k-players belong to T and (ii) w_1 or more type h-players do *not* belong to T. This consideration restricts the numbers of members n_k^T of players of type k in any MWC T with maximum excess. There are three cases, for each of which we show $I(T) - D(T) > 0$:

Case 1:: $n_k^T < w_1$ for all types $1 \le k < b$.

Then the relative weight $\sum_{k \le a} n_k^T \omega_k$ in T of the most over-represented types is less than $\sum_{k \le a} w_1 \omega_k$. So inequality (I) implies $D(T) < \varepsilon$. Since T is a winning coalition, the weight $\sum_{k \ge b} n_k^T \omega_k$ in T of the most under-represented types is greater than $\bar{q} - \sum_{k<b} w_1 \omega_k$. Due to (III) and $\sum_{k \ge b} n_k \omega_k \le 1$ we have $I(T) > \bar{q} - \varepsilon$. So $I(T) - D(T) > \bar{q} - 2\varepsilon \ge 0$.

Case 2:: $n_k^T \ge w_1$ for some $1 \le k \le a$ but $n_h - n_h^T < w_1$ for all $a < h \le t$.[13]

Using that T is a MWC, the relative weight $\sum_{k \le a} n_k^T \omega_k$ in T of the most over-represented types is less than $\bar{q} + \bar{w}_1 - \sum_{k>a} (n_k - w_1)\,\omega_k$ in this case. So inequality (IV) and $\sum_{k \le a} n_k \omega_k \le 1$ imply $D(T) < \bar{q} + \varepsilon$. Moreover, the weight $\sum_{k \ge b} n_k^T \omega_k$ in T of the most under-represented types is greater than

[13]If Case 1 does not apply, there is a smallest index $1 \le k < b$ with $n_k^T \ge w_1$. Assume $k \le a$ first. Because $r_k > r_h$ for all $a < h < t$, the number $n_h - n_h^T$ of h-types outside coalition T is less than w_1: otherwise the indicated replacement would yield a MWC T' with $x^*(T') < x^*(T)$, contradicting the maximum-excess property of T. This is the description of Case 2. The remaining Case 3 involves $a < k < b$ where $r_k > r_h$ for all $b \le h \le t$. Then, analogously, $n_h - n_h^T < w_1$ must hold.

$\sum_{k\geq b}(n_k - w_1)\,\omega_k$ and inequality (II) implies $I(T) > 1-\varepsilon$. So $I(T) - D(T) > 1 - \bar{q} - 2\varepsilon \geq 0$.

Case 3:**:** $n_l^T < w_1$ for all $1 \leq l \leq a$ and $n_k^T \geq w_1$ for some $a < k < b$ but $n_h - n_h^T < w_1$ for all $b \leq h \leq t$.

The relative weight $\sum_{l\leq a} n_l^T \omega_l$ in T of the most over-represented types is then less than $\sum_{l\leq a} w_1\omega_l$. So inequality (I) implies $D(T) < \varepsilon$. Similarly, the total weight of the players of types $b \leq h \leq t$ is greater than $\sum_{h\geq b}(n_h - w_1)\,\omega_h$. Inequality (II) then implies $I(T) > 1-\varepsilon$ and we have $I(T) - D(T) > 1 - 2\varepsilon > \bar{q} - 2\varepsilon \geq 0$.

Recall that $x_k^* \geq w_k$ for all $1 \leq k \leq a$. So for sufficiently small $\sigma > 0$

$$x_k^{**}(\sigma) = \begin{cases} x_k^* - \sigma\omega_k & \text{if } 1 \leq k \leq a, \\ x_k^* & \text{if } a < k < b, \text{ and} \\ x_k^* + \delta\sigma\omega_k & \text{if } b \leq k \leq t \end{cases} \tag{16}$$

with $\delta = \sum_{k\leq a} n_k\omega_k / \sum_{l\geq b} n_l\omega_l > 0$ is an imputation. $x_k^{**}(\sigma)$'s continuity implies existence of $\sigma > 0$ so that no S with $e(S, x^*) < E_1(x^*)$ has maximum excess at $x^{**}(\sigma)$. We fix such a value of σ and write $x^{**} = x^{**}(\sigma)$.

It then suffices to consider coalitions T' with maximum excess at x^* in order to show the contradiction $E_1(x^{**}) < E_1(x^*)$. Such T' has to be winning, and for any MWC $T \subseteq T'$ it must be true that $e(x^*, T) = e(x^*, T') = E_1(x^*)$. Since T and T' both are winning we have $e(x^{**}, T') \leq e(x^{**}, T)$ and

$$E_1(x^{**}) = \max\{e(x^{**}, T)\colon T \text{ is } MWC \text{ and } e(x^*, T) = E_1(x^*)\}. \tag{17}$$

Moreover, for every T on the right-hand side of equation (17)

$$e(x^{**}, T) - E_1(x^*) = e(x^{**}, T) - e(x^*, T) = -\sigma \cdot \big(\underbrace{I(T) - D(T)}_{>0}\big) \cdot \sum_{k\leq a} n_k\omega_k < 0 \tag{18}$$

implies $e(x^{**}, T) < E_1(x^*)$, so that $E_1(x^{**}) < E_1(x^*)$. □

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