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The Potential of the Holler-Packel Value and Index: A New Characterization

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Abstract The Holler-Packel value and (non-normalized) index are given a new characterization by a *potential function*. The Holler-Packel potential of a cooperative game is the total value of all coalitions in which every member makes a positive contribution; restricted to simple games it is simply the number of minimal winning coalitions. New interpretations follow from known equivalence results on existence of a potential function and balanced contributions, path independence, and Shapley blue print properties.

Keywords Holler-Packel value, potential function, public good index, voting power, minimal winning coalitions, opportunities, freedom.

1. Introduction

Hart and Mas-Colell (1988, 1989) introduced the concept of a *potential (function)* to cooperative game theory and applied it to give an elegant characterization of the Shapley value. Based on a slightly modified definition, other authors have demonstrated that other solution concepts for cooperative games, including *simple games* which are frequently applied to analyze power in various economic and political decision bodies, can also be given new characterizations with potential functions. These provide additional interpretations and methods of computation. So far, potentials have been identified for the Banzhaf value (Dragan 1996, Ortmann 1998), the entire class of semivalues (Calvo and Santos 1997), and weighted weak semivalues (Calvo and Santos 1997, 2000). It is known that not all established solutions admit a potential – the nucleolus does not, for example.

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A potential function summarizes a game by a single real number. A given player i's power or expected payoff in the game (as captured by some value with a potential) is simply the difference between this number and the corresponding number of the game obtained by excluding i from the set of players. That very complex games can be meaningfully condensed and players' role in them identified in this way is not only mathematically intriguing; it allows for new interpretations of the considered value.

This note addresses the question of whether the Holler-Packel value and its restriction to simple games admit a potential. In line with Riker's (1962) size principle they consider only coalitions in which every member makes a positive contribution. A pre-formal version of the Holler-Packel value dates back to Luther Martin, a Maryland delegate to the 1787 Constitutional Convention in Philadelphia. It thus has claims to the longest documented history of all power measures (see Felsenthal and Machover 2005). The value also coincides with the member bargaining power measure proposed by Brams and Fishburn (1995) and Fishburn and Brams (1996) when the latter is applied to individual simple games.

We show that the Holler-Packel value indeed admits a potential. It corresponds to the total worth of all coalitions in which every member makes a positive contribution, so-called minimal crucial coalitions. This number provides a summary of players' joint opportunities under Riker's size principle. It can also be viewed as a measure of collective power or even freedom. A given player's value is the respective contribution to it. Complementing the public good interpretation often given to the Holler-Packel value, total opportunities picked up by the potential function increase also for incumbent players as new contributors enter the game. The gains are treated as non-rival in the sense that a given minimal crucial coalition's worth enters all its members' values.

2. Notation, values and indices

A cooperative *game* is a pair (N,v), where $N=\{1,\ldots,n\}$ is a finite set of players with power set $\wp(N)$ and $v:\wp(N)\to\mathbb{R}$ is a *(characteristic) function* which assigns to each subset $S\subseteq N$, called a *coalition*, a real number $v(S)$ with $v(\varnothing)=0$. $v(S)$ is called the *worth* of coalition S. A game for which v only takes values in $\{0,1\}$ is called a *simple game*. A *weighted voting game* is a simple game in which all coalitions with $v(S)=1$ – the *winning coalitions* – can be characterized by a weight w_i for each $i\in N$ and a quota $q>0$ such that $v(S)=1$ if $\sum_{i\in S} w_i \geq q$ and $v(S)=0$ otherwise.

A *solution* $\psi:\boldsymbol{G}\to\mathbb{R}^n$ maps each game $(N,v)\in\boldsymbol{G}$ where $\boldsymbol{G}$ denotes the space of all games or a suitable subclass (e.g. simple games) to a vector $\psi(N,v)$. The real number $\psi_i(N,v)$ is called the *value of player i* in (N,v); it is usually interpreted as either player i's payoff expectation from playing the game or as an indicator of his or her importance or power in the game. A solution ψ on $\boldsymbol{G}$ is often also referred to as a *value*. A value restricted to the class of simple games is also called an *index*.

Values are often defined in terms of players' *marginal contribution* $[v(S)-v(S\setminus i)]$ to the possible coalitions $S\subseteq N$.[1] The class of *probabilistic values* assigns to each player i his or her expected marginal contribution based on a (possibly player-specific subjective) probability distribution over the set of coalitions. Values in which expectations for all players are taken with the same probability measure p and in which $p(S)$ depends at most on the cardinality $s=|S|$ are called *semivalues*. Particularly prominent semivalues are the *Shapley value* φ (Shapley 1953) and the *Banzhaf value* β (Banzhaf 1965, Dubey and Shapely 1979) defined by

$$\varphi_i(N,v)=\sum_{S\subseteq N}\frac{(s-1)!(n-s)!}{n!}[v(S)-v(S\setminus i)]$$

and

$$\beta_i(N,v)=\sum_{S\subseteq N}\frac{1}{2^{n-1}}[v(S)-v(S\setminus i)]$$

respectively. φ is *efficient*, i.e. $\sum_{i\in N}\varphi_i(N,v)=v(N)$, but β is not.

A player i is *crucial* in $S\subseteq N$ if i makes a positive marginal contribution to S. A coalition S in which *every* member $i\in S$ is crucial is called a *minimal crucial coalition (MCC)*. In the context of simple games, MCCs are typically referred to as *minimal winning coalitions*. The set of all MCCs of (N,v) is denoted by $M(N,v)$; the set of all MCCs containing player i is denoted by $M_i(N,v)$. Players not belonging to any MCC are called *dummy players*.

3. Holler-Packel value and indices

The *Holler-Packel value (HPV)* η was first introduced on the class of simple games by Holler (1982), later axiomatized by Holler and Packel (1983), and finally extended to general cooperative games by Holler and Li (1995). It is defined by

$$\eta_i(N,v)=\sum_{S\in M_i(N,v)}v(S).$$

[1] We drop the brackets in $\{i\}$ where there is no danger of confusion.

On the class of *simple games*, it can also be written as

$$\eta_i(N,v) = \sum_{S \in M(N,v)} [v(S) - v(S \setminus i)] = | M_i(N,v) | . \tag{1}$$

Typically, η's *normalized* version, referred to as the *Holler-Packel index (HPI)*,

$$\overline{\eta}_i(N,v) = \frac{\eta_i(N,v)}{\sum_{i \in N} \eta_i(N,v)}$$

is considered in applications. For the purposes of this paper, however, the HPV and non-normalized HPI are of greater interest.

It can be seen from (1) that on the class of weighted voting games, to which both Banzhaf and Holler-Packel values are most commonly applied, β and η only differ (apart from the re-scaling by 2^{1-n}) in that the HPI restricts attention to *minimal* winning coalitions. This is motivated in Holler (1982) by arguing that – e.g. in a world where joining a coalition (endorsing a proposal) is costly – only coalitions where every member matters, i.e. is crucial, 'will be *purposely* formed ('not by sheer luck')' (p. 267, italics added) and should be taken into account. This reflects the classic *size principle* advocated by Riker (1962). The latter is derived from a game-theoretic model discussed in detail by Brams and Fishburn (1995) together with some empirics. Interestingly, the latter authors (also see Fishburn and Brams 1996) derive a measure of so-called *member bargaining power* whose restriction to single simple games coincides with the HPI. The Holler-Packel index is often associated with coalition outcomes interpreted as or relating to public goods and was even introduced as *public good index* (see e.g. Holler 1982).

HPI and HPV have been characterized (for details see Holler and Packel 1983, Holler and Li 1995) by axioms which require from an index or value that it treats players anonymously (i.e. is invariant to permutations of *N*), assigns 0 to dummy players, is additive when a particular sum operation, $v_1 \oplus v_2$, is carried out with games having disjoint sets $M(N,v_k)$, and assigns the full coalition's value $v(S)$ to (at least) one member if this coalition *S* is the unique MCC of the game.

It can easily be seen that, like the Banzhaf value, the HPV is not efficient, i.e. generally $\sum_{i \in N} \eta_i(N,v) \neq v(N)$. Several other properties of the Shapley value are not shared by the HPV either. For example, the HPV (and HPI) can violate weak monotonicity in players' weights for weighted voting games.[2] Whether the possible violation of monotonicity for some particular w and q is a fatal problem or rather an advantage for a voting power index is a matter of debate; see e.g. Brams and Fishburn (1995) and Holler and Napel (2004) for support of the latter view.

The marginal contribution of voter i in any given coalition S is always nondecreasing in weight w_i. It follows that the HPV cannot be expressed as an expected marginal contribution for any probability distribution on $\wp(N)$ which is fixed independently of characteristic function v. In other words, the HPV is *not* a probabilistic value. This does not exclude probabilistic interpretations (see e.g. Widgrén 2001). In particular, the HPI captures players' expected marginal contributions when only MCCs have (equal) positive probability. As in non-cooperative models of multilateral bargaining (e.g. Montero 2002) the probability distribution on $\wp(N)$ is *endogenous* to the players' decision environment.

4. Potential of the Holler-Packel value

Given a game (N,v), let the game $(N \setminus i,v)$ be defined by restricting the domain of the characteristic function, $\wp(N)$, to $\wp(N \setminus i)$. Coalitions involving player i are 'deleted' from the game and all other coalitions $S \subseteq N \setminus i$ simply retain their old worth $v(S)$. If (N,v) is a weighted voting game, $(N \setminus i,v)$ is characterized by the same quota q and weights $w_1,\ldots,w_{i-1},w_{i+1},\ldots,w_n$.[3]

Characterizing a value ψ by a potential function amounts to the provision of a mapping P from the space of all games $\boldsymbol{G}$ or a subclass $G \subseteq \boldsymbol{G}$ closed under the above removal operation to the real numbers. The mapping P must be such that any player i's value $\psi_i(N,v)$ is for any (N,v) equal to the difference between the *potential* $P(N,v)$ of the considered game and the potential $P(N \setminus i,v)$ of the restricted game resulting from dropping player i (letting the remaining players 'play amongst themselves'). In other words, if (and only if) a value $\psi : G \to \mathbb{R}^n$ admits a potential function $P : G \to \mathbb{R}$, one can view and calculate $\psi_i(N,v)$ as i's *marginal contribution*

$$\Delta_i(N,v) \equiv P(N,v) - P(N \setminus i,v)$$

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[2] See Felsenthal and Machover (1998) on this and other 'paradoxes' which the HPI but also the Banzhaf and Shapley-Shubik indices may exhibit.

[3] Slightly uncustomary, our definition of a simple game has *not* required $v(N)=1$. This keeps the space of simple games closed under the considered removal operation.

to the game (N,v), where the latter is evaluated by *P*.[4] Provided a potential exists, then $\psi_i(N,v) = \Delta_i(N,v)$ and ψ's potential satisfies the recursive equation

$$P(N,v) = \frac{1}{n}\left[\sum_{i=1}^{n} \psi_i(N,v) + \sum_{i=1}^{n} P(N \setminus i, v)\right] \tag{2}$$

for all $N \neq \varnothing$ and becomes *uniquely* determined after making the definition

$$P(\varnothing, v) \equiv 0 \ . \tag{3}$$

Hart and Mas-Colell (1988, 1989) were the first to consider the notion of a potential – which has a long tradition in physics (see Ortmann 1998 for a detailed discussion) – in the context of games. They showed that the Shapley value is the unique value which is efficient and admits a potential.[5] More generally, *any* semivalue ψ^{sv} admits a potential (Calvo and Santos 1997). It is

$$P^{\psi^{sv}}(N,v) = \sum_{S \subseteq N} p_s \, v(S)$$

where p_s denotes the probability that a coalition with *s* members is formed.

By construction, a normalized value or index is efficient. Hart and Mas-Colell's characterization of φ as the unique efficient value with a potential therefore implies that the HPI does *not* admit a potential. However, the HPV is neither efficient nor a semivalue. So, the question of whether it admits a potential or not to our knowledge has so far not been answered.

Before we identify the HPV's potential, let us point to several equivalence results that underline its relevance. Namely, a value admits a potential (see Hart and Mas-Colell 1989, Ortmann 1998, Calvo and Santos 1997, Dragan 1999) if and only if it has

- the balanced contribution property (or preserves differences), or
- the path independence property (or is conservative), or

[4] Ortmann (2000) introduced the related notion of a *multiplicative* potential function: a solution ψ admits a multiplicative potential iff there exists a function $P: G \rightarrow \mathbb{R}$ such that $\frac{P(N,v)}{P(N \setminus i, v)} \equiv \psi_i(N,v)$.

[5] Actually, Hart and Mas-Colell originally included efficiency in their definition of a potential function.

- the Shapley blue print property

The property of *balanced contributions* has been defined by Myerson (1980) and intuitively requires that for any two players the gains or losses that one imposes on the other (according to some value ψ) by leaving the game is equal for both. Formally, ψ defined on G satisfies the balanced contribution property iff

$$\forall (N,v)\in G:\forall i,j\in N:\psi_i(N,v)-\psi_i(N\setminus j,v)=\psi_j(N,v)-\psi_j(N\setminus i,v)$$

ψ satisfies *path independence* if, intuitively speaking, one could sequentially 'buy off' players from the game such that they leave one-by-one in exchange for getting 'paid' their value $\psi_i(N',v)$ of the game amongst currently remaining players $N'\subseteq N$, thereby exhausting a *total* amount of money that does *not* depend on the order in which players leave. Formally, denote the set of all permutations $\omega:N\to N$ by $\Omega(N)$ and the set of players preceding i in permutation ω by N_i^{ω}. Then, ψ satisfies path independence iff

$$\forall (N,v)\in G:\forall \omega,\omega' \in\Omega(N):\sum_{i\in N}\psi_i(N_i^{\omega}\cup i,v)=\sum_{i\in N}\psi_i(N_i^{\omega'}\cup i,v)\,.$$

The order-independent total amount spent equals the potential of the game (N,v), namely

$$\begin{aligned}P(N,v)&=P(N,v)-P(\varnothing,v)\\&=\big(P(N,v)-P(N\setminus i_1,v)\big)+\big(P(N\setminus i_1,v)-P(N\setminus\{i_1,i_2\},v)\big)+\ldots\\&\quad+\big(P(i_n,v)-P(\varnothing,v)\big)\\&=\psi_{i_1}(N,v)+\psi_{i_2}(N\setminus i_1,v)+\ldots+\psi_{i_n}(i_n,v)\end{aligned}$$

with $i_k=\omega(k)$ for any given $\omega\in\Omega(N)$.

The *Shapley blue print property*, defined by Dragan (1999), requires a value ψ applied to games (N,v) to equal the Shapley value φ applied to 'related games' (N,v_ψ). In particular, for any given (N,v), the 'related game' (N,v_ψ) – also called *power game* of (N,v) – is defined by

$$v_\psi(S)\equiv\sum_{i\in S}\psi_i(S,v) \tag{4}$$

i.e. the worth of a coalition S in (N,v_ψ) is the sum of the ψ-values of its members in the reduction of the original game to player set S. Clearly,

$(N, v_\psi) \equiv (N, v)$ iff ψ is efficient. The mapping $v \mapsto v_\psi$ allows to connect a value ψ to the many interpretations and results concerning the Shapley value (see e.g. Winter 2002). A value ψ with the Shapley blue print property, $\psi(N, v) \equiv (N, v_\psi)$, may thus be viewed as the 'standard solution' (that is: Shapley's) to coalition formation and distribution problems for a game related to (N, v) in a particular way determined by ψ.

The Banzhaf value and all other semivalues have the properties listed under 1–3. Because these are equivalent to admitting a potential, the same is implied for the HPV by

Proposition 1 $P^\eta : G \to \mathbb{R}$ with $P^\eta(\varnothing, v) = 0$ and otherwise

$$P^\eta(N, v) = \sum_{S \in M(N,v)} v(S) \tag{5}$$

is the potential of the Holler-Packel value η.

Proof Note that

$$\begin{aligned} M(N \setminus i, v) &= \{S \subseteq N \setminus i : j \in S \Rightarrow v(S) > v(S \setminus j)\} \\ &= \{S \subseteq N : j \in S \Rightarrow v(S) > v(S \setminus j)\} \setminus \\ &\quad \{S \subseteq N, i \in S : j \in S \Rightarrow v(S) > v(S \setminus j)\} \\ &= M(N, v) \setminus M_i(N, v). \end{aligned}$$

Therefore

$$\underbrace{\sum_{S \in M(N \setminus i, v)} v(S)}_{P^\eta(N \setminus i, v)} = \sum_{S \in M(N,v) \setminus M_i(N,v)} v(S) = \underbrace{\sum_{S \in M(N,v)} v(S)}_{P^\eta(N,v)} - \underbrace{\sum_{S \in M_i(N,v)} v(S)}_{\eta_i(N,v)}$$

or

$$\eta_i(N, v) = P^\eta(N, v) - P^\eta(N \setminus i, v). \qquad \blacksquare$$

The potential function of HPV's restriction to simple games (N, v), the non-normalized HPI, can also be written as

$$P^\eta(N, v) = |M(N, v)|$$

i.e. a simple game's potential is simply its number of MCCs.

Example As illustration consider the 7-person simple game (N,v) with

$$M(N,v)=\{\{1,2,3,4\},\{3,4,5,6\},\{3,4,5,7\},\{3,5,6,7\},\{4,5,6,7\}\}$$

analyzed by Holler and Li (1995). The potential of the game is $P^{\eta}(N,v)=5$. From

$$M(N\setminus 1,v)=\{\{3,4,5,6\},\{3,4,5,7\},\{3,5,6,7\},\{4,5,6,7\}\}$$

one obtains $P^{\eta}(N\setminus 1,v)=4$, i.e. player 1 contributes a potential of 1 (one MCC) to the game and consequently has a HPV or non-normalized HPI $\eta_1(N,v)=1$. Analogously, $\eta_2(N,v)=1$, $\eta_3(N,v)=\eta_4(N,v)=\eta_5(N,v)=4$, and $\eta_6(N,v)=\eta_7(N,v)=3$.

Now consider, e.g. the coalition $S=\{1,2,3,4\}$. In the restricted game (S,v), the HPV evaluates to $\eta_1(S,v)=\eta_2(S,v)=\eta_3(S,v)=\eta_4(S,v)=1$. So the worth of S in the power game (N,v_η) is $v_\eta(S)=\sum_{i\in S}\eta_i(S,v)=4$. Analogously, one obtains $v_\eta(S',v)=0$ for $S'=S\setminus 1$. So the marginal contribution of player 1 to coalition S in (N,v_η) is $[v_\eta(S)-v_\eta(S\setminus 1)]=4$. Weighted with $\frac{3!3!}{7!}$ this is added to the correspondingly weighted marginal contributions by player 1 to all other coalitions in (N,v_η). The sum total is player 1's Shapley value in game (N,v_η) – which equals player 1's HPV in (N,v), i.e. $\varphi_1(N,v_\eta)=\eta_1(N,v)=1$. ■

Let $\pi(N,v)$ denote the sum of the worths of the MCCs for all players in (N,v), i.e.

$$\pi(N,v)=\sum_{i\in N}\sum_{T\in M_i(N,v)}v(T)$$

We will say that an arbitrary value ψ defined on $G\subseteq \mathbf{G}$ *distributes the worths of MCCs* iff

$$\forall (N,v)\in G:\sum_{i\in N}\psi_i(N,v)=\pi(N,v)$$

This property is similar to efficiency but requires ψ to assign a total value equal to the sum of individual worths experienced in all minimal crucial coalitions. This allows distribution of any given MCC's *full* worth to every member, establishing an analogy to a non-rival good. Clearly the HPV distributes the worths of MCCs. But so does, for example, the value ψ defined by

$$\psi_i(N,v) = \begin{cases} \pi(N,v) & \text{if } i = \min N \\ 0 & \text{otherwise} \end{cases}$$

Matching Hart and Mas-Colell's characterization of the Shapley value (as the unique value which is efficient and admits a potential) and Dragan's and Ortmann's characterization of the Banzhaf value (as the unique value which 'distributes the marginalities' and admits a potential – see Dragan 1996 and Ortmann 1998), the HPV is characterized by

Proposition 2 The Holler-Packel value η is the unique solution which distributes the worths of MCCs and admits a potential.

Proof Let ψ be an arbitrary value with potential P and distribute the worths of MCCs, i.e. $\psi_i(S,v) = \Delta_i(S,v)$ and $\sum_{i\in S} \Delta_i(S,v) = \pi(S,v)$. By equation (2),

$$P(S,v) = \frac{1}{s}[\pi(S,v) + \sum_{i\in S} P(S \setminus i, v)]$$

Consider any coalition $S \subseteq N$ with $|S| = 1$, i.e. $S = \{i\}$ for some $i \in N$. We have $P(S,v) = v(i) = P^{\eta}(S,v)$. Now, $P \equiv P^{\eta}$ (and thus $\psi \equiv \eta$) follows by induction: suppose $P(S,v) = P^{\eta}(S,v)$ for all $S \subseteq N$ with $|S| = s$ and consider $T \subseteq N$ with $|T| = t = s+1$. Then, using that the HPV distributes the values of MCCs,

$$\begin{aligned} P(T,v) &= \frac{1}{t}[\pi(T,v) + \sum_{i\in T} P(T \setminus i, v)] \\ &= \frac{1}{t}[\sum_{i\in T}(P^{\eta}(T,v) - P^{\eta}(T \setminus i, v)) + \sum_{i\in T} P^{\eta}(T \setminus i, v)] \\ &= \frac{1}{t}\sum_{i\in T} P^{\eta}(T,v) = P^{\eta}(T,v) \end{aligned}$$

■

In general, the Shapley value of a game (N,v) may or may not be an element of (N,v)'s *core*. In the former case, the interpretation of (N,v) as a distribution of (expected) payoffs in (N,v) is particularly robust: no coalition could increase own payoffs by breaking away from the grand coalition. Given that η admits a potential, the HPV of (N,v) is just the Shapley value φ of the power game (N,v_{η}) defined by

$$v_\eta(S) \equiv \sum_{i\in S} \eta_i(S,v)$$

In other words, it corresponds to the Shapley division of a surplus whose coalition-specific size equals the involved players' total HPV. Such a surplus division interpretation of the HPV is more relevant if the well-founded Shapley solution of the power game does not conflict with the similarly well-founded core solution, i.e. is coalitionally rational. Indeed we have

Proposition 3 Let $v(S) \geq 0$ for all $S \subseteq N$. Then the Holler-Packel value η of (N,v) lies in the core of (N,v_η).

Proof [6] For any $S \subset T \subseteq N$ and $i \in N$ we have

$$\begin{aligned} M_i(S,v) &= \{R \subseteq S \text{ with } i \in R : k \in R \Rightarrow v(R) > v(R \setminus k)\} \\ &\subseteq \{R \subseteq T \text{ with } i \in R : k \in R \Rightarrow v(R) > v(R \setminus k)\} = M_i(T,v) \end{aligned}$$

Using $v(S) \geq 0$ this implies

$$\eta_i(S,v) = \sum_{T \in M_i(S,v)} v(T) \leq \sum_{T \in M_i(N,v)} v(T) = \eta_i(N,v)$$

Hence for any $S \subseteq N$

$$v_\eta(S) = \sum_{i\in S} \eta_i(S,v) \leq \sum_{i \in S} \eta_i(N,v)$$

i.e. $\eta(N,v)$ is coalitionally rational in (N,v_η). ■

5. Concluding remarks

Hart and Mas-Colell (1989, p. 590) remarked that '[a]lthough the potential is in its essence just a technical tool, it is … a powerful and suggestive one'. Equivalence of its existence with preservation of differences and path independence is a case in point. The potential of a given game can be regarded as a summary of all players' joint opportunities as captured by the corresponding value or index. These opportunities typically increase when a new player joins; the difference of potentials measures this increase and

[6] Alternatively, one can prove that the Holler-Packel power game (N,v_η) is *convex*, implying that its Shapley value is an element of the core.

hence the contribution of the new player. One can start with the empty coalition, then successively draw additional players into the game and see joint opportunities increase exactly by the player's value in the resulting game.

In simple games the Holler-Packel potential can be regarded as the non-weighted number of decision-making opportunities the players collectively have under Riker's size principle. Assuming, first, that the characteristic function v of a simple game just characterizes winning coalitions (not levels of transferable utility) and, second, that players get utility from having opportunities, one may interpret coalitions' worths (and players' values) in the power game v_η as sums of transferable utility. Evaluating players' overall contributions to opportunities in the game under consideration, i.e. determining their Holler-Packel 'power' or 'value' in it, is then equivalent to finding the respective expected utility in the associated power game as predicted by the Shapley value. The total opportunities derived from membership in minimal winning coalitions turn out to expand with non-decreasing marginals when new players are added – or, technically speaking, v_η is convex. Players' predicted utility is therefore in the power game's core, i.e. is also coalitionally rational.

The Holler-Packel potential can be seen as aggregating joint opportunities also in general games. Opportunities arise as in simple games from bringing together a minimal group of players. But they may now vary in quality and quantity depending on which players are members of the coalition. This corresponds to an interpretation of a game's characteristic function v as an index of the quantity and quality of a coalition's opportunities.[7] The potential then captures the total possibilities of performing and benefitting from collective actions via minimal crucial coalitions. It can be interpreted as a measure of quality and quantity-weighted collective power or even freedom in the game under consideration.[8]

Taking players to receive utility from having opportunities in the form of belonging to minimal crucial coalitions, the power game 'distributes' the worths of non-singleton minimal crucial coalitions several times – namely, to all their members. This explains that the sum of players' (non-normalized) Holler-Packel values typically exceeds the worth of the grand coalition. It also adds to the public good interpretation often given for Holler-Packel value and, in particular, Holler-Packel index.

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[7] The restriction to non-negative worth in Prop. 3 then makes sense, whereas it is slightly artificial if v represents utility on an a priori arbitrary scale.

[8] See Holler (2005) and Braham (2006) on the close relationship between freedom and power, and corresponding 'opportunity' and 'exercise' concepts. Also see Laruelle and Valenciano (2005a; 2005b, fn.12) for an explicit link between Coleman's (1971) *power of the collectivity to act* and a generalized potential function.

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